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SMOOTHNESS, SEMI-STABILITY AND ALTERATIONS

by A. J. DE JONG*

CONTENTS

1. Introduction

Let X be a variety over a field k. An *alteration* of X is a dominant proper morphism $X' \to X$ of varieties over k, with $\dim X = \dim X'$. We prove that any variety has an alteration which is regular. This is weaker than resolution of singularities in that we allow finite extensions of the function field $k(X)$. In fact, we can choose X' to be a complement of a divisor with strict normal crossings in some regular projective variety $\overline{X}'$. (For a more precise statement see Theorem 4.1 and Remark 4.2.) If the field k is local, we can find $X' \subset \overline{X}'$ such that $\overline{X}'$ is actually defined over a finite extension $k \subset k'$ and has semi-stable reduction over $\mathcal{O}_{k'}$ in the strongest possible sense, see Theorem 6.5.

Although these results are perhaps not surprising, being consequences of standard conjectures on resolution of singularities, it did surprise the author that they are relatively easy to obtain using results on existence of moduli spaces of stable (pointed) curves.

As an application, we note that Theorem 4.1 implies that for any variety X over a perfect field k, there exist (α) a simplicial scheme $X_\bullet$ projective and smooth over k, (β) a strict normal crossings divisor $D_\bullet$ in $X_\bullet$; we put $U_\bullet = X_\bullet \setminus D_\bullet$, and ($\gamma$) an augmentation $a : U_\bullet \to X$ which is a proper hypercovering of X. (The construction of $X_\bullet$ works

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as in [5, Section 2.3] with Theorem 4.1 instead of Hironaka.) In case k is local, we may assume that the pairs (X_n, D_n) are defined over finite extensions k_n of k and extend to strict semi-stable pairs over $\mathcal{O}_{k_n}$, see 6.3. This should be interpreted as saying that in a suitable category $\mathcal{MM}_k$ of mixed motives over k, any variety X may be replaced by a complex of varieties which are complements of strict normal crossing divisors in smooth projective varieties. The functor $h : \mathcal{V}ar_k \to \mathcal{MM}_k$ should have the property that for any alteration $X' \to X$, the object $h(X)$ corresponding to X may be replaced by the complex $h(X') \to h(X' \times_X X') \to h(X' \times_X X' \times_X X') \to \dots$

We have

$$H^i_{ét}(X \otimes \bar{k}, \mathbf{Q}_\ell) \cong \mathbf{H}^i_{ét}(U_\bullet \otimes \bar{k}, \mathbf{Q}_\ell).$$

The cohomology of each $U_i \otimes \bar{k}$ can be computed by a spectral sequence whose E_1-terms consist of direct sums of cohomology groups occuring in the stratification of $X_i \otimes \bar{k}$ defined by $D_i \otimes \bar{k}$, compare [4, Section 3.2]. As an example, suppose the field k is finite. We see that the eigenvalues of Frobenius on $H^i_{ét}(X \otimes \bar{k}, \mathbf{Q}_\ell)$ occur as eigenvalues of Frobenius on some cohomology group of some smooth projective variety, and hence are Weil numbers; this reproves a result of [3], using [2]. The author had hopes that Theorem 4.1 might imply results on independence of ℓ for étale cohomology, but this seems not automatic. It is clear, by the above, that independence of ℓ for maps $H^i(X_2) \to H^i(X_1)$ given by correspondences between smooth projective varieties X_1, X_2 will imply independence of ℓ results for $H^*(X)$ for arbitrary varieties X. (The best indication in this direction is perhaps the result of Jannsen [13].)

Assume $k = \mathbf{C}$ and $X_\bullet$, $a : U_\bullet \to X$ are as above. We have a canonical isomorphism

$$H^i(X(\mathbf{C}), \mathbf{Q}) \cong \mathbf{H}^i(U_\bullet(\mathbf{C}), \mathbf{Q})$$

of singular cohomology groups. In fact Theorem 4.1 suffices to construct the mixed Hodge structure on $H^i(X(\mathbf{C}), \mathbf{Q})$ as in [5].

Assume $\mathrm{char}(k) = p > 0$ is perfect, let $W = W(k)$ and $K_0 = W \otimes \mathbf{Q}$. We define $H^i_{cris}(X) := \mathbf{H}^i_{cris}(U_\bullet/W) \otimes K_0$. Here we define $\mathbf{H}^i_{cris}(U_\bullet/W)$ as crystalline cohomology of $X_\bullet$ with logarithmic poles along $D_\bullet$. It is clear that this defines a finite-dimensional cristalline cohomology, but unfortunately it is not so clear that the result is independent of the choice of $X_\bullet$, $D_\bullet$ and a. However, it has been shown that Theorem 4.1 implies finite-dimensionality for Berthelot's rigid cohomology with compact supports; furthermore finite-dimensionality for rigid cohomology of smooth varieties follows, and in particular finite-dimensionality for Monsky-Washnitzer cohomology of smooth affines, see [1].

Assume k is a local field and ℓ not equal to the residue characteristic of k. By [21] the ℓ-adic étale cohomology of a strict semi-stable variety X over k is semi-stable: the inertia subgroup $I \subset \mathrm{Gal}(\bar{k}/k)$ acts unipotently and the eigenvalues of Frobenius on the graded parts are Weil numbers. In case X has dimension 2, still in the strict semi-stable case, [21] proves Deligne's conjecture on the purity of the monodromy filtration (see [20, 2.8] for a formulation of this conjecture). Let X be any variety over k; we do not assume X is smooth or proper. The results of [21], together with Theorem 6.5, give that there exists a finite extension $k \subset k'$ such that the $\mathrm{Gal}(\bar{k}/k')$-modules $\mathrm{H}^i_{\text{ét}}(\mathrm{X} \otimes \bar{k}, \mathbf{Q}_\ell)$ are semi-stable for all ℓ (in the sense explained above). If X is smooth and proper over k and has dimension at most 2, then Deligne's conjecture on the purity of the monodromy filtration follows.

Applications to the case where k is local and $\mathcal{O}_k$ has mixed characteristic. We note that Theorem 6.5 proves an implication of the form $\mathrm{C}_{\mathrm{st}} \Rightarrow \mathrm{C}_{\mathrm{DR}}$, see [12] for notation. For, if one proves C_{st} functorially for projective varieties having a strict semi-stable model as in 2.16 (but not necessarily over the same base field), then C_{DR} will follow for an arbitrary smooth projective variety over k. Moreover, if one can prove a conjecture like C_{st} functorially for strict semi-stable pairs as in 6.3, then a conjecture like $\mathrm{C}_{\mathrm{pst}}$ will follow for varieties (singular and/or non proper) over k.

Of course there are other ways to try and apply Theorems 4.1 and 6.5. For example, it has been shown by F. Pop that Theorem 4.1 can be used to prove Grothendieck's conjecture on birational anabelian geometry in characteristic $p > 0$, see [19]. We leave to the reader to find other applications.

We give a short sketch of the argument that proves our results in case X is a proper variety. The idea is to fibre X over a variety Y such that all fibres are curves and work by induction on the dimension of X. After modifying X, we may assume X is projective and normal and we can choose the fibration to be a kind of Lefschetz pencil, where the morphism is smooth generically along any component of any fibre. Next one chooses a sufficiently general and sufficiently ample relative divisor H on X over Y. After altering Y, i.e. we take a base change with an alteration $\mathrm{Y}' \to \mathrm{Y}$, we may assume that H is a union of sections $\sigma_i : \mathrm{Y} \to \mathrm{X}$. The choice of H above gives that for any component of any fibre of $\mathrm{X} \to \mathrm{Y}$, there are at least three sections σ_i intersecting it in distinct points of the smooth locus of $\mathrm{X} \to \mathrm{Y}$. The generic fibre of $\mathrm{X} \to \mathrm{Y}$, together with the points determined by the σ_i is a stable pointed curve. By the existence of proper moduli spaces of stable pointed curves, we can replace Y by an alteration such that this extends to a family $\mathscr{C}$ with sections τ_i of stable pointed curves over Y. An important step is to show that the rational morphism $\mathscr{C} \cdots \to \mathrm{X}$ extends to a morphism, possibly after replacing Y by a modification; this follows from the condition on sections hitting components of fibres above. Thus we see that we may replace X by $\mathscr{C}$. We apply the induction hypothesis to Y and we get Y regular. However, our induction hypothesis is actually stronger and we may assume that the locus of degeneracy of $\mathscr{C} \to \mathrm{Y}$ is a divisor with strict normal crossings. At this point it is clear that the only singularities of $\mathscr{C}$ are given by equations of the type

$$xy = t_1^{n_1} \cdot \ldots \cdot t_d^{n_d}.$$

These we resolve explicitly.

Let us give some instructions to the reader. We advise not to read Section 2: it contains definitions and results, which we assume known in the rest of the paper. (In 2.20, the precise definition of an alteration is given.) In Section 3 we resolve singularities for a family of semi-stable curves over a regular scheme, which is degenerate over a divisor with normal crossings. This we use in Section 4, where we prove the theorem on varieties. Section 5 deals with the problem of altering a family of curves into a family of semi-stable curves. This we use in Section 6, where we do the relative case, i.e. the case of schemes over a complete discrete valuation ring.

In the final two sections we indicate how to refine the method of proof of Theorem 4.1 and Theorem 6.5 to get results where one has additional restraints or works over other base schemes. In Section 7 we prove that our method works (over algebraically closed fields) to get resolution of singularities up to quotient singularities and purely inseparable function field extensions. In fact we deal with the situation where there is a finite group acting. In Section 8 we do the arithmetic case. In particular, we show that any integral scheme X, flat and projective over Spec $\mathbf{Z}$ can be altered into a scheme Y which is semi-stable over the ring of integers in a number field (Theorem 8.2).

In a follow-up of this article the author proves that one can alter any family of curves into a semi-stable family of curves, see [15]. This is stronger than the result of Section 5. In [15] the author deals with group actions as well. Thus in [15] the reader can find a number of results that extend the results (especially Theorems 7.1, 8.2) of this article to (slightly) more general situations. For example it is shown that regular alterations exist of schemes of finite type over two-dimensional excellent base schemes. However, as the methods of that article are more technical, the author feels it is advantageous to the reader to include Sections 7 and 8 in this article, since the methods employed in them are relatively straightforward.

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2. Notation, conventions and terminology

2.1. If X is a scheme and $x \in X$ is a point, then $\kappa(x)$ denotes the residue field of the local ring $\mathcal{O}_{X,x}$. A point is sometimes also considered as a scheme: $x = \operatorname{Spec} \kappa(x)$.

A scheme is *integral* if and only if it is irreducible and reduced. The function field of an integral scheme S is denoted R(S).

2.2. A closed subset Z of the underlying topological space of a scheme X will be confused with the reduced closed subscheme $Z \subset X$ it gives rise to [10, I, 4.6]. In particular an irreducible component $X' \subset X$ will often be considered as a reduced closed subscheme of X.

2.3. In this article a *divisor* D of a scheme X will be a closed subscheme $D \subset X$ regularly embedded of codimension 1, *i.e.* a positive divisor on X [10, IV, 21.1.6 and IV, 21.2.12].

2.4. Let S be a Noetherian scheme. Let $D \subset S$ be a divisor and let $D_i \subset D$, $i \in I$ be its irreducible components (considered as reduced closed subschemes of D or S, see 2.2). We say that D is a *strict normal crossings divisor in* S if *a)* for any $s \in D$ the local ring $\mathcal{O}_{S,s}$ is regular, *b)* D is a reduced scheme, i.e. $D = \bigcup D_i$ (scheme-theoretically) and *c)* for any nonempty subset $J \subset I$, the closed subscheme $D_J = \bigcap_{j \in J} D_j$ is a regular scheme of codimension $\sharp J$ in S [10, IV, 5.1.3].

A divisor D on S is a *normal crossings divisor* if there is a surjective étale morphism $S' \to S$ such that the inverse image of D is a strict normal crossings divisor on S′. In this case there exists a blowing up $\varphi : S' \to S$ in an ideal with support in D such that the reduced inverse image $\varphi^{-1}(D)_{\mathrm{red}}$ is a divisor with strict normal crossings on S′. (In case S is a surface, blow up the singular points of D; in case S is a threefold, first blow up the points where D has three branches, then blow up the strict transform of the curves on S where D has two branches, etc.)

2.5. Let $f : X \to S$ be a morphism of schemes. We write sm(X/S) for the open subscheme of X where f is smooth [10, IV, 17.3].

2.6. Let $f : X \to S$ be a morphism of schemes. We say that f is *generically étale*, if there exists an open dense subscheme $U \subset X$ such that $f|_U$ is étale.

2.7. Let S be a Noetherian scheme and let $f : X \to S$ be a morphism of finite type. *a)* If S is reduced, then there is a dense open subset $U \subset S$ such that $f : X_U \to U$ is flat, cf. [10, IV, 11.3.2]. *b)* If f is proper then the sets $T_d = \{ s \in S \mid \dim f^{-1}(s) \geqslant d \} \subset S$ are closed. *c)* If f is proper dominant, and X and S are integral, then every fibre of f has everywhere dimension at least equal to the dimension of the generic fibre.

2.8. Let $A \to B$ be a local homomorphism of Noetherian complete local rings. Assume A is regular of dimension d, with residue field k. Assume that $\dim B = d + r$ and that $B \otimes_A k$ is formally smooth of dimension r over k. Then B is formally smooth over A.

Proof. — It is easy to reduce to the case $k = \bar{k}$. Then $B \otimes_A k \cong k[[t_1, \ldots, t_r]]$ and by lifting t_i to B we find a surjection $A[[t_1, \ldots, t_r]] \to B$. This cannot have a nontrivial kernel, otherwise $\dim B < d + r$. (Actually, this proof works as soon as A is a domain.) Q.E.D.

2.9. Let k be a field. A *variety* X *over* k is a separated scheme of finite type over Spec k which is irreducible and reduced, i.e. integral. This definition does not agree with the definition of a variety as given in some of the textbooks on algebraic geometry, e.g. Hartshorne's Algebraic Geometry, where one assumes that X is geometrically integral over k.

2.10. If X is a variety over k and the function field $k(\mathrm{X})$ is separable over k, then the structural morphism $p : \mathrm{X} \to \mathrm{Spec}(k)$ is smooth over a nonempty open subscheme of X. If k is perfect, then $\mathrm{sm}(\mathrm{X}/\mathrm{Spec}(k)) = \mathrm{Reg}(\mathrm{X})$, the regular locus of X. This is not true if k is not perfect, even in case $k \subset k(\mathrm{X})$ is separable. Example: $\mathrm{V}(y^p - x^2 - \alpha) \subset \mathbf{A}^2_k$, where $\alpha \in k$ is not a p-th power and the characteristic of k is p.

2.11. Let k be a field and let $\mathrm{X} \subset \mathbf{P}^n_k$ be a closed subscheme, with the following properties: *a)* X has pure dimension $d < n$; *b)* there exists a dense open subscheme $\mathrm{V} \subset \mathrm{X}$ which is geometrically reduced over k.

Let $k \subset k'$ be a finite separable extension, let $p \in \mathbf{P}^n(k')$, $p \notin \mathrm{X}_{k'}$. Consider the projection morphism

$$\mathrm{pr}_p : \mathrm{X}_{k'} \to \mathbf{P}^{n-1}_{k'}$$

of $\mathrm{X}_{k'}$ to the projective space of lines $\ell \subset \mathbf{P}^n$ passing through p. This is a finite morphism as all fibres are finite (being equal to $\ell \cap \mathrm{X}_{k'} \neq \ell$) and X is projective. We claim there exists a nonempty open subset $\mathrm{U} \subset \mathbf{P}^n_{k'}$ such that for $p \in \mathrm{U}(k')$ the morphism pr_p has the following property:

(α) if $\dim \mathrm{X} = d < n - 1$, then $\mathrm{pr}_p : \mathrm{X}_{k'} \to \mathrm{pr}_p(\mathrm{X}_{k'})$ is birational;

(β) if $\dim \mathrm{X} = d = n - 1$, then $\mathrm{pr}_p : \mathrm{X}_{k'} \to \mathbf{P}^d_{k'}$ is finite étale over a nonempty open subset of $\mathbf{P}^d_{k'}$ (equivalently: pr_p is generically étale, see 2.6).

These statements are geometric, hence it suffices to prove them over a separable algebraic closure of k. (Any nonempty open subscheme $\mathrm{U}^{\mathrm{sep}} \subset \mathbf{P}^n_{k^{\mathrm{sep}}}$ contains an open subscheme of the form $\mathrm{U} \otimes_k k^{\mathrm{sep}}$, with $\mathrm{U} \subset \mathbf{P}^n_k$ nonempty.) The conditions may be checked component by component, hence we may assume X irreducible. By condition *b)* there exists a nonempty open subscheme V′ of X which is smooth over k. Choose a k-valued point $q \in \mathrm{V}'(k)$; this is possible as k is separably closed. In case (α) consider all lines ℓ through q, meeting X only in q and transversally in q, i.e. $\ell \cap \mathrm{X} = q$ scheme theoretically. We may choose $p \neq q$ on any such line ℓ, since $\mathrm{pr}_p^{-1}(\ell) = q$ will imply that $\mathrm{pr}_p : \mathrm{X} \to \mathrm{pr}_p(\mathrm{X})$ is birational. Clearly such p sweep out a nonempty open subset of $\mathbf{P}^n_k$. In case (β) consider all lines ℓ through q which intersect X transversally at q. The morphism $\mathrm{pr}_p : \mathrm{X} \to \mathbf{P}^d_k$ for $p \neq q$ on such a line ℓ is unramified at the point $q \in \mathrm{X}$. Therefore the stalk at q of the sheaf $\Omega^1_{\mathrm{X}/\mathbf{P}^d}$ vanishes, and pr_p is generically étale. (One could also use 2.8.) The result follows.

2.12. We will denote (complete) discrete valuation rings with the letter R (or R′ and sometimes $\mathcal{O}$). Usually, the residue field of R is written k and a uniformizer is denoted by π, or sometimes π_{R}. A *morphism of (complete) discrete valuation rings* $\mathrm{R} \to \mathrm{R}'$ will refer to a local ring homomorphism such that π_R is not mapped to zero. The ramification index $e = e(R'/R)$ of such a morphism is simply the valuation of the element π_R in the valuation of R'. We note that a morphism of complete valuation rings $R \to R'$ is formally smooth (see [10, Ch. 0, 19.3.1], we take the $\mathfrak{m}$-adic topology on R) if and only if $e(R'/R) = 1$ and the residue field extension $k \subset k'$ is separable. If we have morphisms of discrete valuation rings $R \to R' \to R''$, and $R \to R''$ is formally smooth, then so is $R \to R'$.

A scheme S is called a *trait* if it is isomorphic to a scheme Spec R, where R is a complete discrete valuation ring. The generic point of a trait S is denoted η, and the special point is denoted s. We write π_S for a uniformizer in $\mathcal{O}_S$. A *morphism of traits* is a morphism $S' \to S$ corresponding to a morphism of complete discrete valuation rings $R \to R'$ as above. Such a morphism is said to be a finite extension of traits if the extension $R \subset R'$ is finite.

2.13. *Lemma.* — *Let* R *be an excellent discrete valuation ring; put* $S = \operatorname{Spec} R$. *Let* X *be a normal integral* S*-scheme which is flat of finite type over* S *and let* $\xi \in X$ *be a generic point of the special fibre* X_s *of* X. *Put* $\mathcal{O} = \mathcal{O}_{X,\xi}$. *There exists an extension* $R \subset R'$ *of discrete valuation rings* $R \subset R'$ *such that*

(i) $Q(R) \subset Q(R')$ *is finite, and*

(ii) *the algebra* $\mathcal{O}' = (\mathcal{O} \otimes_R R')^{\mathrm{norm}}$ *(the normalization of the reduction of* $\mathcal{O} \otimes_R R'$*) is formally smooth over* R', *i.e. the localizations* $\mathcal{O}_i$ *of* $\mathcal{O}'$ *at its maximal ideals are discrete valuation rings with* $e(\mathcal{O}_i/R') = 1$ *and the field extensions* $k' \subset \mathcal{O}_i/\pi_{R'}\mathcal{O}_i$ *are separable.*

For any further extension $R' \subset R''$ *for which* (i) *holds, the result of* (ii) *also holds.*

Proof (Faltings). — We reduce to the case $\dim X/S = 1$. We argue by induction on $\dim X/S$; so assume that $\dim X/S > 1$. Choose an element $t \in \mathcal{O}_{X,\xi}$ such that the image of t in $\kappa(\xi)$ is transcendental over k. Thus we get (after shrinking X) a dominant morphism $f\colon X \to \mathbf{A}^1_S$ such that $f(\xi) = \zeta$ is the generic point of $\mathbf{A}^1_s$.

By induction hypothesis, we find an extension of discrete valuation rings $\mathcal{O}_{\mathbf{A}^1_S,\zeta} \subset \mathcal{O}'$ with $K(t) \subset Q(\mathcal{O}')$ finite and such that

$$\mathcal{O}' \subset (\mathcal{O}_{X,\xi} \otimes_{\mathcal{O}_{\mathbf{A}^1_S,\zeta}} \mathcal{O}')^{\mathrm{norm}}$$

is formally smooth. We have $\mathcal{O}' \cong \mathcal{O}_{Y,\zeta'}$ for some integral normal S-scheme Y, with $Y \to \mathbf{A}^1_S$ finite, dominant and ζ' a generic point of Y_s. Hence by induction, we find an extension of discrete valuation rings $R \subset R'$, such that $Q(R) \subset Q(R')$ is finite and

$$R' \subset (\mathcal{O}_{Y,\zeta'} \otimes_R R')^{\mathrm{norm}}$$

is formally smooth. We have homomorphisms

$$\begin{aligned}\mathcal{O}_{X,\xi} \otimes_R R' &\to \mathcal{O}_{X,\xi} \otimes_{\mathcal{O}_{\mathbf{A}^1,\zeta}} \mathcal{O}_{Y,\zeta'} \otimes_R R' \\ &\to (\mathcal{O}_{X,\xi} \otimes_{\mathcal{O}_{\mathbf{A}^1,\zeta}} \mathcal{O}_{Y,\zeta'})^{\mathrm{norm}} \otimes_{\mathcal{O}_{Y,\zeta'}} (\mathcal{O}_{Y,\zeta'} \otimes_R R')^{\mathrm{norm}} =: B.\end{aligned}$$

The algebra B is formally smooth over $(\mathcal{O}' \otimes_{\mathrm{R}} \mathrm{R}')^{\mathrm{norm}}$ by base change and $(\mathcal{O}' \otimes_{\mathrm{R}} \mathrm{R}')^{\mathrm{norm}}$ is formally smooth over R'. By transitivity we get that B is formally smooth over R'. In this situation we may, by extending R', assume that $\mathrm{Q}(\mathrm{R}) \subset \mathrm{Q}(\mathrm{R}')$ is normal. Let D be the decomposition group of R' in the Galois group. There is a nontrivial $\mathcal{O}$-homomorphism $(\mathcal{O} \otimes_{\mathrm{R}} \mathrm{R}')^{\mathrm{norm}} \to \mathrm{B}$, and we conclude that one of the localizations $\mathcal{O}_i$ is formally smooth over R' (see 2.12). However, D acts transitively on the set of such localizations $\mathcal{O}_i$, whence the result for all i.

The case $\dim \mathrm{X}/\mathrm{S} = 1$. We may assume that the component $\mathrm{X}(\xi)$ of X_s with generic point ξ has genus at least 2; by this we mean that any irreducible component of $\mathrm{X}(\xi) \otimes \overline{\kappa(s)}$ has genus at least 2. If not, then replace X by the normalization X' in a function field extension given by an equation of the form $y^\ell - f = 0$ for some rational function f on X. We can take $f \in \mathcal{O}_{\mathrm{X},\xi}$ such that $f|_{\mathrm{X}(\xi)}$ is a nonvanishing rational function having at least three simple poles ($\ell \neq \mathrm{char}\, k$ and $\ell \geqslant 5$). There will be a unique point $\xi' \in \mathrm{X}'$ lying over ξ such that $\mathcal{O}_{\mathrm{X},\xi} \subset \mathcal{O}_{\mathrm{X}',\xi'}$ is finite étale and $g(\mathrm{X}'(\xi')) \geqslant 2$. The result for $\mathcal{O}_{\mathrm{X}',\xi'}$ implies the result for $\mathcal{O}_{\mathrm{X},\xi}$.

We may assume that $\mathrm{X} \to \mathrm{S}$ is projective. Let $\mathrm{K} = \mathrm{Q}(\mathrm{R})$. For some finite extension $\mathrm{K} \subset \mathrm{K}''$, the normalization of the reduction of $\mathrm{X}_\eta \otimes \mathrm{K}''$ is a union of smooth projective curves of genus at least 2. After a finite extension $\mathrm{K}'' \subset \mathrm{K}'$ these curves all get stable reduction over a discrete valuation ring $\mathrm{R}' \supset \mathrm{R}$ with $\mathrm{Q}(\mathrm{R}') = \mathrm{K}'$. Thus there exists a scheme $\mathrm{X}' \to \operatorname{Spec} \mathrm{R}'$ which is a finite disjoint union of stable curves over Spec R and such that $\mathrm{X}' \otimes \mathrm{K}'$ is equal to the normalization of the reduction of $\mathrm{X} \otimes \mathrm{K}'$. Let X'' be the normalization of the reduction of $\mathrm{X} \otimes \mathrm{R}'$. Thus X'' is birational to the scheme X' and there is a blow up $\mathrm{X}''' \to \mathrm{X}'$ which dominates X'':

$$\begin{array}{ccccccc}
\mathrm{X}' & \longleftarrow & \mathrm{X}''' & \longrightarrow & \mathrm{X}'' & \longrightarrow & \mathrm{X} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} \mathrm{R}' & \xleftarrow{\mathrm{id}} & \operatorname{Spec} \mathrm{R}' & \xrightarrow{\mathrm{id}} & \operatorname{Spec} \mathrm{R}' & \longrightarrow & \operatorname{Spec} \mathrm{R}.
\end{array}$$

For any component X_i'' of X_s'' mapping onto $\mathrm{X}(\xi)$, there is a unique component X_i''' of X_s''' lying over it. Since $g(\mathrm{X}_i''') = g(\mathrm{X}_i'') \geqslant g(\mathrm{X}(\xi)) \geqslant 2$, it is not contracted to a point on X'. We get a component X_i' of X_s'. Let ξ_i'', resp. ξ_i''', resp. ξ_i' be the generic points of the components X_i'', etc. Thus, as the schemes X'', X''' and X' are birational, we have

$$\mathcal{O}_{\mathrm{X}',\xi_i'} \cong \mathcal{O}_{\mathrm{X}''',\xi_i'''} \cong \mathcal{O}_{\mathrm{X}'',\xi_i''}.$$

Thus we see that $\mathcal{O}_{\mathrm{X}'',\xi_i''}$ is formally smooth over R', as X' is stable over R'. It is easy to check that these are the discrete valuation rings $\mathcal{O}_i$ occuring in the statement of the lemma. Q.E.D.

2.14. *Remark.* — There are similar statements to be found in the literature. See for example [8], where a result is proved in the rigid analytic setting; see [7] for lowering the invariant e of an extension of discrete valuation rings (and a counterexample to too optimistic conjectures); see [14, Section 6.3] for a method to deal with the residue extensions in the complete mixed characteristic case.

2.15. Let S be a trait. An S-*variety* X is an irreducible, reduced and separated scheme X, flat and of finite type over S. The condition of flatness is equivalent (in view of the fact that X is integral) to the condition that X_η is a nonempty variety over the field $\kappa(\eta)$. Let $f: X \to S$ be the structural morphism. We write $f^{-1}(s)$, X_s or $X \otimes \kappa(s)$ for the (scheme theoretic) fibre of X at the point $s \in S$. We write $f^{-1}(\{s\})$ for the set-theoretic inverse image of the point $s \in S$ (but note that we sometimes view $f^{-1}(\{s\})$ as a reduced closed subscheme of X, see 2.2). We remark that $X_s = V(\pi_S)$ is a (principal) divisor on the scheme X.

2.16. Let S be a trait and let X be an S-variety. Let X_i, $i \in I$ be the irreducible components of X_s. Put $X_J = \bigcap_{j \in J} X_j$ (scheme-theoretic intersection), for a nonempty subset J of I. We say that X is *strictly semi-stable over* S if the following properties hold:

a) X_η is smooth over $\kappa(\eta)$,

b) X_s is a reduced scheme, i.e. $X_s = \bigcup X_i$ scheme-theoretically,

c) for each $i \in I$, X_i is a divisor on X, and

d) for each nonempty $J \subset I$, the scheme X_J is smooth over $\kappa(s)$ and has codimension $\sharp J$ in X.

We remark that these conditions imply that X is a regular scheme, see below.

(Local description of strictly semi-stable S-varieties.) Let $x \in X_s$ be a point of the special fibre. Suppose that x lies on the components $X_1, \ldots, X_r$ and not on the other components of X_s. Let $t_i \in \mathcal{O}_{X,x}$ be an element such that $V(t_i) = X_i \cap \operatorname{Spec} \mathcal{O}_{X,x}$. Consider the completion A of $\mathcal{O}_{X,x}$ and let $\overline{B} = A/(t_1, \ldots, t_r)$. By *d)*, $\overline{B}$ is a formally smooth $\kappa(s)$-algebra. Hence we can find a complete local R-algebra B, formally smooth over R, which lifts $\overline{B}$. Further, since $\overline{B}$ is formally smooth over $\kappa(s)$, we can find a $\kappa(s)$-algebra homomorphism $\overline{B} \to A/\pi A$, which is a section of $A/\pi A \to \overline{B}$. Thus it is clear that $A/\pi A \cong \overline{B}[[t_1, \ldots, t_r]]/(t_1 \cdot \ldots \cdot t_r)$. We can lift this to a surjection $B[[t_1, \ldots, t_r]] \to A$ (again using formal smoothness) and we see that

$$A \cong B[[t_1, \ldots, t_r]]/(t_1 \cdot \ldots \cdot t_r - \pi).$$

(To get rid of the unit in front of π, change one of the t_i by a unit.) We conclude that a neighbourhood of x is smooth over the scheme $\operatorname{Spec} R[t_1, \ldots, t_r]/(t_1 \cdot \ldots \cdot t_r - \pi)$.

We define *semi-stable* S-*varieties* (i.e. not necessarily strictly so) by requiring that the situation étale locally looks as described above. We will not make use of this definition.

We remark that if X is proper over S, then *b)*, *c)* and *d)* imply *a)*. Furthermore, if $\kappa(s)$ is perfect, then *b)*, *c)* and *d)* are equivalent to the statement: X_s is a divisor with strict normal crossings on X, see 2.10. Of course the concept of strict semi-stability is most useful if X is proper over S.

2.17. Let S be an integral Noetherian scheme. A *modification* S′ of S is an integral scheme S′, together with a proper birational morphism $\varphi : S' \to S$. The center of the modification is the closed subset of S over which φ is not an isomorphism. A composition of modifications is a modification.

2.18. Let $f : X \to S$ be a morphism of finite type, with S Noetherian and integral. Let $\psi : S' \to S$ be a modification. We consider the diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \times_S S' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ S' \arrow[r, "\mathrm{id}"] & S' \arrow[r, "\psi"] & S. \end{tikzcd}$$

Here X' is the closed subscheme of $X \times_S S'$ given by dividing the $\mathcal{O}_{S'}$-torsion out of $\mathcal{O}_{X \times_S S'}$. We remark that X' may be empty, even if $S' = S$. The morphism $f' : X' \to S'$ is called the *strict transform* of f (with respect to ψ). There is an obvious transitivity property in case we have a second modification $\psi' : S'' \to S'$.

There exists a nonempty open subscheme $U \subset S$ such that $X_U \to U$ is flat, see 2.7. Clearly, $X'|_{\psi^{-1}(U)} \cong X \times_U \psi^{-1}(U)$ and X' is the schematic closure of this in $X \times_S S'$. Thus if X is integral and dominates S, then $X' \to X$ is a modification as well. Finally, if X'' is closed in $X \times_S S'$, flat over S' and equal to $X \times_S S'$ over a nonempty open part of S', then it equals the schematic closure of $X \times_U \psi^{-1}(U)$ and hence equals the strict transform of X, i.e. $X'' = X'$.

2.19. We recall some arguments from [22, p. 36-37]. Let $f : X \to S$ be a projective morphism of Noetherian schemes. We assume S integral. Let $U \subset S$ be a nonempty open subscheme over which f is flat; such exist, see 2.7. We claim that there exists a modification $\psi : S' \to S$ with center in $S \backslash U$ such that the strict transform $f' : X' \to S'$ of f is flat.

Choose a relatively ample line bundle $\mathcal{L}$ on X over S. Let $P(t)$ be the Hilbert polynomial of $\mathcal{L} \otimes \kappa(u)$ on X_u for $u \in U$. Consider the functor $\mathrm{Hilb}^P_{X/S} : (\mathcal{S}ch/S)^0 \to \mathcal{S}et$ which associates to T over S the set of closed subschemes $Z \subset X \times_S T$ of finite presentation over $X \times_S T$ and flat over T, such that $\mathcal{L}|_Z$ has Hilbert polynomial P on all fibres of $Z \to T$. The theory of Hilbert schemes, see [9], applies and gives that $\mathrm{Hilb}^P_{X/S}$ is representable by a scheme Hilb projective over S. The S-morphism $U \to \mathrm{Hilb}$, which we have by construction, extends to a morphism $S' \to \mathrm{Hilb}$, where $S' \to S$ is a modification with center in $S \backslash U$. Indeed, we can take S' to be the schematic closure of $U \hookrightarrow \mathrm{Hilb}$. This gives $Z \subset X \times_S S'$, flat over S', which is the strict transform of X, see 2.18.

We remark that for the existence of a modification $S' \to S$ such that the strict transform of f is flat, it suffices that f is locally (in S) projective. Locally, one then finds a modification $U' \to U$ as above, these then glue since they solve a universal problem, see [22, p. 37]. Thus we can, using the arguments described above, deal with proper morphisms $f : X \to S$ which have the property that all the fibres have dimension $\leqslant 1$. (Such a morphism is automatically étale locally projective.) This will suffice for the applications we have in mind. Anyway, we can deal with arbitrary morphisms f of finite type (S Noetherian) by invoking [22, Theorem 5.2.2].

2.20. Let S be a Noetherian integral scheme. An *alteration* S′ of S is an integral scheme S′, together with a morphism $\varphi : S' \to S$, which is dominant, proper and such that for some nonempty open $U \subset S$, the morphism $\varphi^{-1}(U) \to U$ is finite. (This last condition is equivalent to the condition $\dim S = \dim S'$, at least if these are finite.) If $\varphi : S' \to S$ is an alteration there is a nonempty largest open subscheme $U \subset X$ such that $\varphi^{-1}(U) \to U$ is finite and flat; the complement of U is called the center of the alteration. A composition of alterations is an alteration. One can define the strict transform of a morphism $f: X \to S$ with respect to alterations as in 2.18. We remark that an alteration $S' \to S$ is generically étale if and only if the (finite) extension of function fields $R(S) \subset R(S')$ is separable.

2.21. Let S be a scheme. A *semi-stable curve* X *over* S is a flat proper $f: X \to S$ of finite presentation, such that all geometric fibres are connected curves having at most ordinary double points as singularities. Let $\mathrm{Sing}(f) \subset X$ be the closed subscheme defined by the first Fitting ideal of the sheaf $\Omega^1_{X/S}$. The morphism $\mathrm{Sing}(f) \to S$ is finite, unramified and of finite presentation.

2.22. We say that a semi-stable curve X over a field k is *split* if *a)* all the irreducible components of X are geometrically irreducible and smooth over k and *b)* all singular points of X are k-rational. Let $f: X \to S$ be a semi-stable curve over S. We say that f is *split* or that X is a *split semi-stable curve over* S if for any $s \in S$ the fibre X_s is a split semi-stable curve over $\kappa(s)$. We remark that the pullback by $S' \to S$ of a split semi-stable curve is a split semi-stable curve over S′.

2.23. (Local description of (split) semi-stable curves.) Let $f: X \to S$ be a semi-stable curve with S Noetherian. Consider a point $x \in \mathrm{Sing}(f)$ and let $s = f(x)$. The extension $k = \kappa(s) \subset k' = \kappa(x)$ is finite separable, as $\mathrm{Sing}(f) \to S$ is finite unramified. Let B be the complete local ring of X at x and let A be the complete local ring of S at s. Choose a finite étale extension of local rings $A \to A'$ realizing the residue field extension $k = A/\mathfrak{m}_A \subset k' = A'/\mathfrak{m}_{A'}$. As B is henselian and $B/\mathfrak{m}_B \cong k'$, the ring homomorphism $A \to B$ factors through A′; we get $A \to A' \to B$.

By assumption of semi-stability we have that $B/\mathfrak{m}_A B \cong k'[[u, v]]/(q)$, where $q = q(u, v) = a_0 u^2 + a_1 uv + a_2 v^2$ is a quadratic form with coefficients in k' and non-vanishing discriminant. Choose an arbitrary lift

$$Q = Q(u, v) = A_0 u^2 + A_1 uv + A_2 v^2 \in A'[u, v] \text{ of } q(u, v).$$

By flatness of B over A′, we see that $B \cong A'[[u, v]]/(Q - h)$ for some $h \in \mathfrak{m}_{A'}.A'[[u, v]]$. Rechoosing the coordinates $u, v \in B'$ appropriately, using that $\mathrm{discr}(q) \neq 0$, we see that we may assume $h \in A'$. (One may also prove this by showing that the minimal versal deformation space of the singularity $k'[[u, v]]/(q)$ is one-dimensional.)

We remark that in this case (i.e. $h \in A'$) the trace of $\mathrm{Sing}(f)$ on the scheme $\mathrm{Spec}\, B$ is given by the ideal $(u, v) \subset B$. This lies over the closed subscheme of $\mathrm{Spec}\, A$ given by the element $\mathrm{N}m_{A'/A}(h) \in \mathfrak{m}_A \subset A$, the map being finite étale.

If f is a split semi-stable curve, then $k = k'$ and the quadratic form q splits over k. The last assertion follows as there are two components passing through x, both defined over k. Thus we may choose $q = uv$ and $Q = uv$. The complete local ring of X at x is

$$B \cong A[[u, v]]/(uv - h)$$

for some $h \in A$.

2.24. A general reference for this subsection is [16] (see also [11] for the case of genus 0). Fix $g \geqslant 0$, $n \geqslant 3$ and let $\overline{\mathcal{M}}_{g,n}$ denote the algebraic stack over $\mathbf{Z}$ classifying stable n-pointed curves of genus g. The open substack $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ classifies smooth n-pointed curves. Let ℓ be a prime number at least 3. Let

$${}_\ell\mathcal{M}_{g,n} \to \mathcal{M}_{g,n}[1/\ell] = \mathcal{M}_{g,n} \times_{\mathrm{Spec}\, \mathbf{Z}} \mathrm{Spec}\, \mathbf{Z}[1/\ell]$$

be the finite étale cover given by trivializing the ℓ-torsion of the Jacobian of the universal genus g curve over $\mathcal{M}_{g,n}[1/\ell]$. We remark that ${}_\ell\mathcal{M}_{g,n} = {}_\ell\mathrm{M}_{g,n}$ is a scheme (usual arguments). Finally, let

$${}_\ell\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}[1/\ell] = \overline{\mathcal{M}}_{g,n} \times_{\mathrm{Spec}\, \mathbf{Z}} \mathrm{Spec}\, \mathbf{Z}[1/\ell]$$

be the normalization of $\overline{\mathcal{M}}_{g,n}[1/\ell]$ in the function field of ${}_\ell\mathcal{M}_{g,n}$. We remark that ${}_\ell\overline{\mathcal{M}}_{g,n} = {}_\ell\overline{\mathrm{M}}_{g,n}$ is a projective scheme over $\mathrm{Spec}\, \mathbf{Z}[1/\ell]$, compare [6]. By pullback from $\overline{\mathcal{M}}_{g,n}[1/\ell]$ we get a "universal" stable n-pointed curve of genus g over ${}_\ell\overline{\mathrm{M}}_{g,n}$.

We can get a scheme $\overline{\mathrm{M}}$ projective over $\mathrm{Spec}\, \mathbf{Z}$, with a "universal" curve over it by taking distinct primes $\ell_1, \ell_2 \geqslant 3$ and putting $\overline{\mathcal{M}}$ equal to the normalization of $\overline{\mathcal{M}}_{g,n}$ in the function field of ${}_{\ell_1\ell_2}\mathcal{M}_{g,n}$. Then $\overline{\mathcal{M}} = \overline{\mathrm{M}}$ is a projective scheme over $\mathbf{Z}$ with a finite dominant morphism $\overline{\mathrm{M}} \to \overline{\mathcal{M}}_{g,n}$. Compare [6].

3. Semi-stable curves and normal crossings divisors

3.1. Let S be an excellent regular scheme and let $D \subset S$ be a divisor with strict normal crossings. (Recall that this means that $D = \bigcup_{i \in I} D_i$, each D_i is a divisor on S and for any nonempty subset $J \subset I$ the scheme $D_J = \bigcap_{j \in J} D_j$ is a regular scheme of codimension $\sharp J$ in S.) Assume we have a semi-stable curve $f : X \to S$ smooth over $S \backslash D$. We remark that the singular locus $\mathrm{Sing}(X)$ of the scheme X is contained in $\mathrm{Sing}(f)$.

3.2. *Lemma. — There exists a projective modification* $\varphi_1 : X_1 \to X$ *with the following properties*:

(i) *The center of* φ_1 *lies in* $\mathrm{Sing}(X)$.
(ii) X_1 *is a semi-stable curve over* S, *smooth over* $S \backslash D$.
(iii) $\mathrm{Sing}(X_1)$ *has codimension at least three in* X_1.
(iv) *If the curve* X *is split semi-stable* (2.22), *then the curve* X_1 *is split semi-stable over* S.

3.3. Let us describe the local situation at a point $x \in \mathrm{Sing}(X)$. Put $s = f(x) \in S$. Let $A \to A' \to B$ be as in 2.23, and choose an isomorphism $B \cong A'[[u, v]]/(Q - h)$, with $h \in A'$ (2.23). Suppose that s lies in the components $D_1, \ldots, D_r$ of D but not in any other component. Let $t_i \in \mathcal{O}_{S,s}$ be an element such that $V(t_i) = D_i \cap \mathrm{Spec}\, \mathcal{O}_{S,s}$. Note that the elements $t_1, \ldots, t_r$ form a regular system of parameters of $\mathcal{O}_{S,s}$, hence also of A and A'. The singular locus of f traced on Spec B maps isomorphically to the closed subscheme $V(h) \subset \mathrm{Spec}\, A'$. By assumption we have $V(h) \subset V(t_1 \cdot \ldots \cdot t_r)$. Therefore we see that

$$h = \varepsilon \cdot t_1^{n_1} \cdot \ldots \cdot t_r^{n_r}, \qquad \varepsilon \in (A')^*$$

with $n_i \geqslant 0$ and $\Sigma n_i \geqslant 2$ (if $\Sigma n_i = 1$, then the point x is regular on X). We change Q into $\varepsilon^{-1} Q$. Thus we have

$$B \cong A'[[u, v]]/(Q - t_1^{n_1} \cdot \ldots \cdot t_r^{n_r}).$$

We remark that B is the completion of the algebra

$$B' = A'[u, v]/(Q - t_1^{n_1} \cdot \ldots \cdot t_r^{n_r})$$

at the maximal ideal $\mathfrak{m}_A B' + (u, v) B'$. For questions which are not sensitive to completion, we may compute using the algebra B'.

Furthermore, in the split case we may assume $A = A'$ and $Q(u, v) = uv$, see 2.23.

3.4. Let T be an irreducible component of Sing(X) of codimension 2 in X. (Note that $\mathrm{codim}(\mathrm{Sing}(X), X) \geqslant 2$, either by the formulae above, or by noting that X is normal.) Since $T \subset \mathrm{Sing}(X) \subset \mathrm{Sing}(f)$, the morphism $T \to D$ is finite unramified, hence for reasons of dimensions T maps onto a component D_1 of D and the map is finite unramified. As D_1 is regular, we conclude that T is regular too.

Further, if $x \in T \subset X$ and we write the complete local ring B of X in x as in 3.1, then we see that the complete local ring of T at x corresponds to the quotient map

$$B \cong A'[[u, v]]/(Q - t_1^{n_1} \cdot \ldots \cdot t_r^{n_r}) \to A'/t_1 A'.$$

The integer n_1 must be $\geqslant 2$, otherwise X is regular along the generic point of T, in contradiction to our assumption. This integer is independent of the choice of $x \in T$, i.e. it is an invariant n_T of the codimension 2 irreducible component T of Sing(X).

We write $\varphi : X' \to X$ for the blowing up of X in the ideal sheaf of T, and let $f' : X' \to S$ be the structural map. We claim that (i) the center of φ lies in Sing(X), (ii) X' is a semi-stable curve over S, smooth over $X \setminus D$, (iii) the invariant n_T has dropped and (iv) if X is split, then so is X'. More precisely, (iii) means the following: Let T' be an irreducible component of Sing(X). There exists at most one such irreducible component $T'' \subset \mathrm{Sing}(X')$ lying above T'; we have $n_{T'} = n_{T''}$, unless $T' = T$ in which case we have $n_{T''} = n_T - 2$.

Clearly, the lemma follows from the claim, by repeatedly blowing up components of the singular locus of codimension 2 in X and induction on the numbers n_T.

We remark that completion and blowing up commute in a suitable manner, so that it suffices to compute the blow up of Spec B′ (3.3) in the ideal (u, v, t_1). Also we only treat the split case, as the non-split case reduces to this one after a finite étale base extension. This blow up is covered by three affine charts (corresponding to the coordinates u, v, t_1); we just give the affine algebras and the reader may read of properties (i)-(iv) from this.

Chart: "$u \neq 0$". — Here we get

$$A[u, v, v', t_1']/(v - uv', t_1 - ut_1', v' - u^{n_1-2}(t_1')^{n_1} t_2^{n_2} \cdot \ldots \cdot t_r^{n_r}).$$

For the convenience of the reader we give the special fibre intersected with this chart; it is the spectrum of the ring

$$k[u, v, v', t_1']/(v - uv', ut_1', v') \cong k[u, t_1']/(ut_1')$$

at least if some $n_i > 0$, $i \neq 1$. This scheme is smooth over k, except at the maximal ideal (u, t_1'). Thus the algebra is regular along $(f')^{-1}(s)$ by 2.8 and f' smooth, except perhaps for the point $u = t_1' = 0$. However, the equation $t_1 - ut_1'$ shows that our algebra is regular in this point also. (It is rather clear that all components of the special fibre are smooth, defined over k, and that the singular points are k-rational.)

Chart: "$v \neq 0$". — By symmetry this is the same as above.

Chart: "$t_1 \neq 0$". — Here we get

$$A[u, v, u', v']/(u - t_1 u', v - t_1 v', u' v' - t_1^{n_1-2} t_2^{n_2} \cdot \ldots \cdot t_r^{n_r}).$$

Again the situation is rather clear. The "new" component T′ lying over T is given by $u' = v' = t_1 = 0$, unless $n_1 = 2, 3$, then T′ lying over T does not exist. Clearly, n_T has dropped by 2. This proves Lemma 3.2.

3.5. (Local description of the case $\mathrm{codim}(\mathrm{Sing}(X), X) \geqslant 3$.) Looking at the equations $Q - t_1^{n_1} \cdot \ldots \cdot t_r^{n_r}$ for a point $x \in \mathrm{Sing}(X)$ as in 3.3, we see that we must have $n_i \in \{0\ 1, \}$. Thus B looks like $A'[[u, v]]/(Q - t_1 \cdot \ldots \cdot t_\mu)$ for some $2 \leqslant \mu \leqslant r$ and D at s is defined by $t_1 \cdot \ldots \cdot t_r = 0$. We remark that this implies that Sing(X) has pure codimension three in X. (In the equations above X is singular along $u = v = t_1 = t_2 = 0$.) Let $\mathrm{Sing}(X) = \bigcup E_\alpha$ be the decomposition into irreducible components of Sing(X). Each E_α maps in a finite étale manner to an irreducible component of some $D_i \cap D_j$, $i \neq j$, hence E_α is a regular scheme.

3.6. *Proposition.* — *Let* $f : X \to S$ *be a split semi-stable curve smooth over* $S \setminus D$, *as in* 3.1. *There exists a projective modification* $\varphi_1 : X_1 \to X$ *with the following properties*:

(i) *The center of* φ_1 *lies in* Sing(X).

(ii) X_1 *is a split semi-stable curve over* S, *smooth over* $S \setminus D$.

(iii) *The scheme* X_1 *is regular.*

Proof. — By Lemma 3.2 we may assume $\operatorname{codim}(\operatorname{Sing}(X), X) \geqslant 3$. Let D_i be an irreducible component of D and let $E \subset X$ be an irreducible component of $f^{-1}(D_i)$. We are going to blow up X in the ideal sheaf of the closed subscheme E. Let $\varphi : X' \to X$ be this blowup.

If $x \in E$ is a regular point of X, then φ will be an isomorphism at x as E can then be defined by one equation. Hence consider $x \in \operatorname{Sing}(X)$. By the above the complete local ring B of X at x looks like

$$B \cong A[[u, v]]/(uv - t_1 \cdot \ldots \cdot t_\mu),$$

where D in A is given by $t_1 \cdot \ldots \cdot t_r = 0$ $(r \geqslant \mu \geqslant 2)$. If D_i corresponds to $t_i = 0$ for some $i > \mu$, then E is given by the principal ideal $(t_i) \subset B$, hence φ is an isomorphism at x. If not, say D_i is given by $t_1 = 0$; then E will be given by the ideal (u, t_1) or the ideal (v, t_1). This follows from the fact that f is split: The morphism $\chi : \operatorname{Spec} B/(u, v, t_1) \to X$ maps into $\operatorname{Sing}(f)$ and $\operatorname{Im}(\chi)$ dominates D_i. (Since $\operatorname{Spec} B/(u, v, t_1) \cong \operatorname{Spec} A/(t_1)$ dominates D_i as $A/(t_1)$ is the completion of $\mathcal{O}_{D_i, s}$.) Thus there are two components E_1, E_2 of $f^{-1}(D_i)$ each containing $\operatorname{Im}(\chi)$, by the assumption that f is split applied to the fibre over the generic point of D_i. We have $x \in E_1 \cap E_2$, hence we get two components through x, one given by (u, t_1) the other given by (v, t_1) and one of them is E.

Therefore we study the blow up of the ring $B' = A[u, v]/(uv - t_1 \cdot \ldots \cdot t_\mu)$ in the ideal (u, t_1). There are two charts.

Chart: "$u \neq 0$". — We get the algebra

$$A[u, v, t_1']/(t_1 - ut_1', v - t_1' \cdot t_2 \cdot \ldots \cdot t_\mu) \cong A[u, t_1']/(ut_1' - t_1).$$

Chart: "$t_1 \neq 0$". — We get the algebra

$$A[u, v, u']/(u - t_1 u', u' v - t_2 \cdot \ldots \cdot t_\mu) \cong A[v, u']/(u' v - t_2 \cdot \ldots \cdot t_\mu).$$

We observe that $f' : X' \to S$ is again a split semi-stable curve, smooth over $S \setminus D$. Furthermore, the modification $X' \to X$ has center in $\operatorname{Sing}(X)$ as observed above. Therefore, its exceptional locus has codimension at least 2 (since the fibres have dimension at most 1). Thus the number of irreducible components of $f^{-1}(D)$ is equal to the number of irreducible components of $(f')^{-1}(D)$, a bijection given by taking inverse image under φ. Moreover, the component $\varphi^{-1}(E)$ of $(f')^{-1}(D)$ has become a divisor on X', whereas this property also holds for the components of $(f')^{-1}(D)$ corresponding to components of $f^{-1}(D)$ having this property on X. Thus repeatedly blowing up components of $f^{-1}(D)$, we arrive at the situation where all components of $f^{-1}(D)$ are divisors.

However, the discussion above shows that at any point x of X where the invariant μ is at least 2, there are at least two components E_1, E_2 of $f^{-1}(D)$, passing through x, which are not divisors at x. Thus $\mu = 1$ everywhere, i.e. X is regular. Q.E.D.

4. Varieties

4.1. *Theorem.* — *Let* X *be a variety over a field* k *and let* $Z \subset X$ *be a proper closed subset. There exist an alteration*

$$\varphi_1 : X_1 \to X$$

and an open immersion $j_1 : X_1 \to \overline{X}_1$ *such that*

(i) $\overline{X}_1$ *is a projective variety and is a regular scheme, and*
(ii) *the closed subset* $j_1(\varphi_1^{-1}(Z)) \cup \overline{X}_1 \setminus j_1(X_1)$ *is a strict normal crossings divisor in* $\overline{X}_1$.

If k *is perfect then the alteration* φ_1 *may be chosen to be generically étale.*

4.2. *Remark.* — The construction of $\overline{X}_1$ below gives that the morphism $\overline{X}_1 \to \operatorname{Spec} k$ factors through $\operatorname{Spec} k_1$, with $k \subset k_1$ finite, such that $\overline{X}_1$ is geometrically irreducible and smooth over k_1. If k is perfect, then $\overline{X}_1$ is smooth over k, see 2.10.

4.3. We argue by induction on $d = \dim X$. The case $\dim X = 0$ is all right. (And so is the case $\dim X = 1$, as we see by taking X_1 to be the normalization of X. However, this does not give the result of Remark 4.2. To get this see 4.5.)

4.4. *Strategy of proof.* — We proceed step by step, each time reducing the theorem to a case where we have additional restaints, numbered (i), (ii), etc., on the pair (X, Z). For example, if $\varphi : X' \to X$ is an alteration, put $Z' = \varphi^{-1}(Z)$. If we can prove the theorem for the pair (X′, Z′), then the theorem will follow for the pair (X, Z). Here we assume that φ is generically étale in case k is perfect. Usually, the properties (i), (ii), etc., will be preserved in this process.

4.5. Let $\bar{k}$ be an algebraic closure of k. Let $X^\heartsuit$ be an irreducible component of the scheme $X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$, and let $Z^\heartsuit$ be the inverse image of Z in $X^\heartsuit$. Suppose we can find $\varphi_1^\heartsuit : X_1^\heartsuit \to X^\heartsuit$ and $j_1^\heartsuit : X_1^\heartsuit \hookrightarrow \overline{X}_1^\heartsuit$ over $\bar{k}$ as in the theorem. There exists a finite extension k_1 of k contained in $\bar{k}$ such that $X^\heartsuit$, $X_1^\heartsuit$, $\overline{X}_1^\heartsuit$, $\varphi_1^\heartsuit$ and $j_1^\heartsuit$ exist over k_1. Thus we have X', X_1, $\overline{X}_1$ and j_1 over k_1 and $\varphi_1' : X_1 \to X'$, such that these give rise to $X^\heartsuit$, $X_1^\heartsuit$, $\overline{X}_1^\heartsuit$, $\varphi_1^\heartsuit$ and $j_1^\heartsuit$ over $\bar{k}$. If we put $\varphi_1 : X_1 \to X$ equal to the composition of φ_1' with the natural morphism $X' \hookrightarrow X \otimes k_1 \to X$ then the quadruple $(X_1, \overline{X}_1, \varphi_1, j_1)$ is a solution to the problem posed in the theorem. (Note that if $\varphi_1^\heartsuit$ is generically étale then so is φ_1', and if k is perfect, then $X' \to X$ will be generically étale too.) Therefore it suffices to prove the theorem under the additional hypothesis:

(i) The field k is algebraically closed.

4.6. We apply Chow's lemma to the variety X. This gives a modification $\varphi : X' \to X$ such that X′ is quasi-projective over k. Put $Z' = \varphi^{-1}(Z)$. As remarked in 4.4, and by 4.5 we reduce to the case where we have (i) and

(ii) X is quasi-projective.

4.7. Assume (i) and (ii). Let $j : X \to \overline{X}$ be an open immersion of X into a projective variety $\overline{X}$ over the field k. Let $\overline{Z} = j(Z) \cup \overline{X} \setminus X$. Suppose that $\overline{\varphi}_1 : \overline{X}_1 \to \overline{X}$, $\overline{j}_1$ solves the problem for the pair $(\overline{X}, \overline{Z})$. Clearly, $\overline{X}_1$ is a proper variety, hence $\overline{j}_1$ is an isomorphism. Put $X_1 = \overline{\varphi}_1^{-1}(j(X))$, $\varphi_1 = \overline{\varphi}_1|_{X_1}$ and $j_1 : X_1 \to \overline{X}_1$ the inclusion morphism. Note that $j_1(\varphi_1^{-1}(Z)) \cup \overline{X}_1 \setminus X_1 = \overline{\varphi}_1^{-1}(\overline{Z})$ and that φ_1 is an alteration. We have reduced the problem to the case where we have (i) and

(iii) X is projective.

4.8. Assume (i) and (iii). Let $\varphi : X' \to X$ be the blowing up in the ideal sheaf of Z (considered as a reduced closed subscheme, see 2.2). Thus φ is a modification of X. Note that $Z' = \varphi^{-1}(Z)$ is the reduction of a divisor $D' \subset X'$ (see 2.3). We apply the procedure explained in 4.4 and we reduce to the case where we have (i), (iii) and the property:

(iv) There exists a divisor $D \subset X$ such that Z is the support of D.

4.9. Assume (iii) and (iv). We may enlarge Z on X. Indeed, suppose that Z' is a closed subset of X containing Z and that we can solve the problem for the pair (X, Z') as in the theorem. Then the closed subset $\varphi_1^{-1}(Z)$ will have pure codimension 1 in the variety $X_1 = \overline{X}_1$ and will be contained in the strict normal crossings divisor $\varphi_1^{-1}(Z')$. It follows that $\varphi_1^{-1}(Z)$ is a strict normal crossings divisor.

4.10. The property (iv) is preserved if we apply any alteration $\varphi : X' \to X$ as in 4.4. The property (i) is trivially preserved, and (iii) holds as long as we only take projective alterations φ. In particular, taking φ equal to the normalization morphism we may assume in addition to (i)-(iv) that we have

(v) X is a normal variety.

4.11. *Lemma. — Suppose that the pair* (X, Z) *over* k *satisfies* (i)-(iv). *There exist a modification* $\varphi : X' \to X$ *and a morphism* $f : X' \to \mathbf{P}^{d-1}$ *of varieties having the following properties :*

(i) *There exists a finite subset* $S \subset \mathrm{Reg}(X)$, *consisting of closed points, disjoint from* Z, *such that* $\varphi : X' \to X$ *is equal to the blowing up of* X *in* S.

(ii) *a) All fibres of* f *are equidimensional of dimension* 1 *and nonempty.*

b) The smooth locus of f *is dense in all fibres of* f.

c) Let $Z' = \varphi^{-1}(Z)$, *which we consider as a reduced closed subscheme of* X'. *The morphism* $f|_{Z'} : Z' \to \mathbf{P}^{d-1}$ *is finite and étale over an open subscheme of* $\mathbf{P}^{d-1}$.

If X *is normal, i.e. if* (X, Z) *satisfies* (v), *then we may choose* φ *and* f *such that in addition we have*

d) At least one fibre of f *is smooth.*

Proof. — There exists a finite morphism

$$\pi : X \to \mathbf{P}^d$$

which is étale over an open subset of $\mathbf{P}^d$, and such that $\pi|_Z : Z \to \pi(Z)$ is birational. To construct such a π, we choose an embedding $X \hookrightarrow \mathbf{P}^N$, and we let π be the composition of projection morphisms as in 2.11, adapted to Z and X.

Let $B \subset \mathbf{P}^d$ be the branch locus of π, more precisely, the complement of B is the locus over which π is étale. Note that $\pi(Z)$ is a reduced closed subscheme of $\mathbf{P}^d$, equidimensional of dimension $d-1$. Therefore, for a general point $p \in \mathbf{P}^d$, $p \notin B \cup \pi(Z)$, the morphism $\mathrm{pr}_p : \pi(Z) \to \mathbf{P}^{d-1}$ is étale over a nonempty open subscheme of $\mathbf{P}^{d-1}$. See 2.11.

Choose p and put

$$X' = \{ (x, \ell) \in X \times \mathbf{P}^{d-1} \mid \pi(x) \in \ell \}.$$

It is easy to see that X' equals the blowing up of X in the finite set $\pi^{-1}(p)$, which is contained in the regular locus of X (since $p \notin B$) and disjoint from Z (since $p \notin \pi(Z)$). The fibres of the morphism

$$f = \mathrm{pr}_2 : X' \to \mathbf{P}^{d-1}$$

are the schemes $\pi^{-1}(\ell)$ (scheme theoretic inverse image, i.e. $\pi^{-1}(\ell) = \ell \times_{\mathbf{P}^d} X$). Since $\pi^{-1}(\ell) \to \ell$ is finite, we see that $\pi^{-1}(\ell)$ has dimension at most 1. On the other hand, ℓ is given locally by $d-1$ equations, hence $\pi^{-1}(\ell)$ has at every point dimension at least 1, as X has pure dimension d. Any component of $\pi^{-1}(\ell)$ is finite over ℓ, hence contains one of the points of $\pi^{-1}(\{p\})$. Thus it suffices to show that f is smooth along the exceptional fibres E_i of $X' \to X$. Locally in the étale topology, f along E_i looks like $\widetilde{\mathrm{pr}}_p : \widetilde{\mathbf{P}}^d \to \mathbf{P}^{d-1}$ along the exceptional fibre of the blowing up $\widetilde{\mathbf{P}}^d \to \mathbf{P}^d$ of $\mathbf{P}^d$ in p. This proves (ii) *a)* and *b)*. Assertion (ii) *c)* is clear as $Z' \cong Z \to \pi(Z) \to \mathbf{P}^{d-1}$ is generically étale by construction.

To prove the last assertion, we go back to the method whereby we constructed π. We see that $\pi = \mathrm{pr}_L$ is the linear projection of X onto $\mathbf{P}^d$ with center in some linear variety $L \subset \mathbf{P}^N$ of dimension $N - d - 1$ in general position. The space L together with the point p, define a linear subvariety $L' \subset \mathbf{P}^N$ of dimension $N - d$ and the fibres of f are the intersections $X \cap H$, where H varies over the linear subvarieties of dimension $N - d + 1$ containing L'. By the usual Bertini arguments we see that choosing L and p general gives that there is at least one H such that the curve $X \cap H$ is smooth over k.
Q.E.D.

4.12. Assume (i)-(v) and apply 4.11. This gives $\varphi : X' \to X$ and $f : X' \to \mathbf{P}^{d-1}$. Note that X' is normal also. We remark that f, having one nonsingular fibre and $\mathbf{P}^{d-1}$ being nonsingular imply that f is smooth over a nonempty open part of $\mathbf{P}^{d-1}$, see 2.8. (This could have been seen in the proof of 4.11 as well.) Let $X' \to Y' \to \mathbf{P}^{d-1}$ be the Stein factorization of f. Note that $Y' \to \mathbf{P}^{d-1}$ is (finite) étale, in view of property (ii) *b)* of the lemma (cf. [18]). (The reader may circumvent this result by replacing $\mathbf{P}^{d-1}$ by Y'.)

We conclude that $Y' = \mathbf{P}^{d-1}$, hence all fibres of f are geometrically connected. By replacing (X, Z) by (X', Z'), see 4.4 and 4.10, we may assume we have (i)-(v) and the following property:

(vi) There exists a morphism $f: X \to Y$ of projective varieties such that:

(vi) *a)* All fibres are nonempty, geometrically connected and equidimensional of dimension 1.

(vi) *b)* The smooth locus of f is dense in all fibres.

(vi) *c)* The generic fibre of f is smooth.

(vi) *d)* The morphism $f|_Z : Z \to Y$ is finite and generically étale.

4.13. *Lemma.* — *Suppose $f: X \to Y$ is a morphism of projective varieties over an algebraically closed field k satisfying* (vi) *a) and b). There exists a divisor $H \subset X$ such that*

(i) *$f|_H : H \to Y$ is finite and generically étale, and*

(ii) *for all geometric points $\bar{y} \in Y$ and any irreducible component C of $X_{\bar{y}} = f^{-1}(\bar{y})$ we have*

$$\# sm(X/Y) \cap C \cap H \geqslant 3.$$

Here we count the number of points of the underlying set, i.e. not with multiplicities.

Proof. — We fix a natural number $n \in \mathbf{N}$. Let $\mathscr{L}$ be a very ample line bundle on X and let

$$i : X \to \mathbf{P} = \mathbf{P}(\Gamma(X, \mathscr{L}^{\otimes n}))$$

be the projective embedding associated to $\mathscr{L}^{\otimes n}$. Note that for any irreducible curve $C \subset X$, the curve $i(C) \subset \mathbf{P}$ is not contained in any linear subspace of dimension $n - 1$. Indeed, since $\mathscr{L}$ is very ample the image of the map $\Gamma(X, \mathscr{L}) \to \Gamma(C, \mathscr{L}|_C)$ contains a base point free pencil $V \subset \Gamma(C, \mathscr{L}|_C)$. The map $\mathrm{Sym}^n V \to \Gamma(C, \mathscr{L}^{\otimes n}|_C)$ has rank at least $n + 1$. Hence $\Gamma(X, \mathscr{L}^{\otimes n}) \to \Gamma(C, \mathscr{L}^{\otimes n}|_C)$ has rank at least $n + 1$, which proves the assertion.

We write $\mathbf{P}^\vee$ for the dual projective space of hyperplanes in $\mathbf{P}$. Consider

$$T = \{ (H, y) \in \mathbf{P}^\vee \times Y \mid \dim f^{-1}(y) \cap H = 1 \} \subset \mathbf{P}^\vee \times Y.$$

Note that T is Zariski closed in $\mathbf{P}^\vee \times Y$: it is the locus over which the morphism

$$\begin{array}{rcl} \{ (H, x) \in \mathbf{P}^\vee \times X \mid x \in H \} & \xrightarrow{\pi} & \mathbf{P}^\vee \times Y \\ (H, x) & \mapsto & (H, f(x)) \end{array}$$

has fibres of dimension 1. (Note: all fibres of π have dimension at most 1. Use 2.7.) The morphism $\mathrm{pr}_2 : T \to Y$ has geometric fibres $\mathrm{pr}_{\bar{y}}$ which can be described as follows:

$$\mathrm{pr}_2^{-1}(\bar{y}) = \bigcup_{C \subset f^{-1}(\bar{y})} \{ H \in \mathbf{P}^\vee \otimes \kappa(\bar{y}) \mid i(C) \subset H \}.$$

The union is over all irreducible components C of $f^{-1}(\bar{y})$. But $i(C)$ is a curve in $\mathbf{P} \otimes \kappa(\bar{y})$ not contained in any linear space of dimension $n - 1$. Hence

$$\mathrm{codim}(\mathrm{pr}_2^{-1}(\bar{y}), \mathbf{P}^\vee \otimes \kappa(\bar{y})) \geqslant n.$$

Thus

$$\dim T \leqslant \dim Y + \dim \mathbf{P}^\vee - n.$$

We see that taking n large enough gives T large codimension in $\mathbf{P}^\vee \times Y$, hence the closed subset $\mathrm{pr}_1(T)$ of $\mathbf{P}^\vee$ has large codimension in $\mathbf{P}^\vee$. In particular, $\mathrm{pr}_1(T) \neq \mathbf{P}^\vee$.

Let $y \in Y(k)$ be a closed point. Consider

$$U = \left\{ H \in \mathbf{P}^\vee(k) \;\middle|\; \begin{array}{l} H \notin \mathrm{pr}_1(T) \\ H \cap f^{-1}(y) \subset \mathrm{sm}(X/Y) \\ H \cap f^{-1}(y) \text{ is a reduced scheme} \end{array} \right\}.$$

Here $f^{-1}(y)$ denotes the scheme-theoretic fibre. The second condition means that H avoids the finite set (cf. *b)*) which is the complement of $\mathrm{sm}(X/Y)$ in $f^{-1}(y)$ and the third condition that H is transversal to $f^{-1}(y)$ in all the points of intersection. This gives a nonempty open set of $\mathbf{P}^\vee$. Thus by the above (if n is large enough) $U \subset \mathbf{P}^\vee$ is nonempty open. Take $H' \in U$ and put $H = X \cap H'$. The morphism $f|_H : H \to Y$ is quasi-finite hence finite (as $H' \notin \mathrm{pr}_1(T)$). At each of the intersection points $x \in f^{-1}(y) \cap H$ we have $\mathcal{O}^\wedge_{X,x} \cong \mathcal{O}^\wedge_{Y,y}[[t]]$, since f is smooth at x. Further H is defined by $(h) \subset \mathcal{O}^\wedge_{X,x}$ with $h \in t + \mathfrak{m}_y \mathcal{O}^\wedge_{X,x}$, as $\mathcal{O}_{H \cap f^{-1}(y),x} \cong k$. Therefore $f|_H : H \to Y$ is finite étale over a neighbourhood of y in Y. (Actually, we could also have used 2.8 to see this, at least if Y is, say, normal.) Any component of H dominates Y in view of dimensions, hence $f|_H$ is generically étale.

We claim that there exists an open neighbourhood $U \subset Y$ of y such that (ii) holds for geometric points of U. Indeed, any open neighbourhood $U \subset Y$ such that *a)* $f: H \cap f^{-1}(U) \to U$ is finite étale and *b)* $H \cap f^{-1}(U) \subset \mathrm{sm}(X/Y)$ works. We can find U with *a)*, see above, and *b)* will follow after shrinking U, as $H \cap f^{-1}(y) \subset \mathrm{sm}(X/Y)$ and an étale morphism is open. (Actually, *a)* implies *b)*.) Choose any geometric point $\bar{y}$ of U and a component C of $X_{\bar{y}}$. Note that $C \cap H$ consists of exactly $\deg C$ points (over $\kappa(\bar{y})$) in view of *a)*. Furthermore, these are all contained in $\mathrm{sm}(X/Y)$ in view of *b)*. The degree of C in $\mathbf{P}$ is at least n, hence if $n \geqslant 3$ we get our claim.

If $U \neq Y$, take a closed point $y' \in Y \setminus U$ and choose $H' \subset X$, which has property (i) and satisfies (ii) over an open neighbourhood U' of y'. Then $H \cup H' \subset X$ satisfies (i) and (ii) over $U \cup U'$. Continuing like this we get the result by Noetherian induction.

Q.E.D.

4.14. Assume the pair (X, Z) satisfies (i)-(v) and (vi) *a)-d)*. We apply 4.13 to the morphism $f: X \to Y$ of (vi). This gives $H \subset X$. It suffices to prove the theorem for the pair $(X, Z \cup H)$, see 4.9. Note that this pair also satisfies (i)-(v), (vi) *a)-d)* and

(vi) *e)* For all geometric points $\bar{y}$ of Y and any irreducible component C of $X_{\bar{y}}$ we have

$$\# \mathrm{sm}(X/Y) \cap C \cap Z \geqslant 3.$$

4.15. Assume (i)-(v), (vi) *a)-e)*. In the sequel we will consider projective alterations $\psi : Y' \to Y$, which are generically étale. Consider the diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & Y' \times_Y X \arrow[r, "pr_2"] \arrow[d, "pr_1"] & X \arrow[d, "f"] \\ Y' \arrow[r, "id"] & Y' \arrow[r, "\psi"] & Y \end{tikzcd}$$

Here X′ is the reduction of the scheme $Y' \times_Y X$, i.e. f' is the strict transform of f with respect to the alteration ψ, cf. 2.20 and 2.18. Put $Z' = pr_2^{-1}(Z)_{red}$ considered either as a closed subscheme of $Y' \times_Y X$ or of X′. It is easy to see that the morphism f' satisfies (vi) *a)-c)*; from this we conclude that X′ is irreducible. Thus $\varphi : X' \to X$ is a projective alteration which is generically étale. Conditions (i), (iii) and (iv) are all right for the pair (X′, Z′), but (v) may fail, for example if Y′ is not normal. Finally, (vi) *d)* is trivial to verify for $f'|_{Z'}$ and (vi) *e)* holds since $\varphi^{-1}(\mathrm{sm}(X/Y)) \subset \mathrm{sm}(X'/Y')$. As before it is clear that it suffices to prove the theorem for the pair (X′, Z′).

4.16. Assume (i)-(iv), (vi) *a)-e)*. Let $Z = \bigcup_{i \in I} Z_i$ be the decomposition into irreducible components of Z. Choose a finite separable Galois extension $k(Y) \subset L$ such that $k(Z_i)$ may be embedded over $k(Y)$ into L for all i; this is possible as the field extensions $k(Y) \subset k(Z_i)$ are finite separable by (vi) *d)*. Let Y′ be the normalization of Y in the field L, then $\psi : Y' \to Y$ is a (finite) generically étale alteration of Y. Constructing (X′, Z′) as in 4.15, we see that $Z' = Z'_1 \cup \ldots \cup Z'_n$ with $Z'_i \to Y'$ finite and birational. (Any component of Z′ dominates Y′ as Z′ has pure codimension 1 in X′ in view of (iv).) Thus $Z'_i \to Y'$ is an isomorphism as Y′ is normal. It follows that we may assume the following property in addition to (i)-(iv), (vi) *a)-e)*:

(vi) *f)* There are sections $\sigma_i : Y \to X$, $i = 1, \ldots, n$ of f such that $Z = \bigcup \sigma_i(Y)$.

We note that (vi) *f)* is also preserved by alterations as in 4.15.

4.17. Assume (i)-(iv), (vi) *a)-f)*. We define an open subscheme $U \subset Y$ by the formula

$$U = \{y \in Y \mid X_y \text{ is smooth over } y \text{ and } \sigma_i(y) \neq \sigma_j(y) \text{ for } i \neq j\}.$$

By (vi) *c)* we have $U \neq \emptyset$. Let g denote the genus of $f^{-1}(y)$ for $y \in U$. In view of (vi) *e)* we have $n \geq 3$ (with n as in (vi) *f)*), hence $(X_U, \sigma_1|_U, \ldots, \sigma_n|_U)$ is a stable n-pointed curve of genus g over U. This defines a 1-morphism

$$U \to \mathcal{M}_{g,n}$$

of U into the algebraic stack classifying smooth stable n-pointed curves of genus g, see 2.24 for notation and results. Choose $\ell \geq 3$ prime to the characteristic of k. Let

$$U' \subset U \times_{\mathcal{M}_{g,n}} {}_\ell M_{g,n}$$

be an irreducible component; it is finite étale over U, nonempty since ℓ is prime to the characteristic of k. Put Y' equal to the closure of

$$\mathrm{Im}(U' \to Y \times {}_{\ell}\overline{M}_{g,n}) \subset Y \times {}_{\ell}\overline{M}_{g,n}.$$

It is clear that Y' is a projective variety over k and that $\psi : Y' \to Y$ is an alteration which is generically étale. The smooth stable n-pointed curve $(X_U, \sigma_1|_U, \dots, \sigma_n|_U) \times_U U'$ extends to a stable n-pointed curve over Y', see 2.24. (The 1-morphism $U' \to U \to \mathcal{M}_{g,n}$ extends to $Y' \to {}_{\ell}\overline{M}_{g,n} \to \overline{\mathcal{M}}_{g,n}[1/\ell] \to \overline{\mathcal{M}}_{g,n}$ by construction.)

Replacing Y by Y' and X by X' as in 4.15 we reduce to a case in which (i)-(iv), (vi) *a)-f)* hold and

(vi) *g)* There exist a stable n-pointed curve $(\mathcal{C}, \tau_1, \dots, \tau_n)$ over Y, a nonempty open subscheme $U \subset Y$ and an isomorphism $\beta : \mathcal{C}_U \to X_U$ mapping the section $\tau_i|_U$ to the section $\sigma_i|_U$.

Again we remark that (vi) *g)* is preserved by operations as in 4.15, by putting $\mathcal{C}' = \mathcal{C} \times_Y Y'$, etc.

4.18. We want to prove that the rational map β extends to a morphism of $\mathcal{C}$ into X, perhaps after replacing Y by a modification. Since we will need a similar statement later on we prove the result in a slightly more general situation.

Suppose we are given a proper morphism $f: X \to S$ of integral excellent schemes, with sections $\sigma_1, \dots, \sigma_n$ satisfying the following properties:

a) All fibres of f are nonempty, geometrically connected and equidimensional of dimension 1.

b) The smooth locus of f is dense in all fibres.

c) The generic fibre of f is smooth.

e) For all geometric points $\bar{s}$ of S and any irreducible component C of $X_{\bar{s}}$ we have for $Z = \bigcup \sigma_i(S)$ that

$$\# \operatorname{sm}(X/Y) \cap C \cap Z \geqslant 3.$$

g) There exist a stable n-pointed curve $(\mathcal{C}, \tau_1, \dots, \tau_n)$ over S, a nonempty open subscheme $U \subset S$ and an isomorphism $\beta : \mathcal{C}_U \to X_U$ mapping the section $\tau_i|_U$ to the section $\sigma_i|_U$.

Let us define T as the closure of Γ_β in the scheme $\mathcal{C} \times_S X$. Note that T is integral, as it is the closure of the integral scheme $\Gamma_\beta \cong \mathcal{C}_U$ (flat, geometrically reduced and connected fibres over U and irreducible generic fibre). Let $S' \to S$ be a modification and apply the reasoning of 4.15. This gives a new set of data X', σ_i', $\mathcal{C}'$, τ_i', β' over the scheme S' satisfying the properties *a)-c)*, *e)* and *g)*. We remark that X' is the strict transform of X, see 2.18. Similarly, the closed subscheme $T' \subset \mathcal{C}' \times_{S'} X' \subset (\mathcal{C} \times_S X) \times_S S'$ is the strict transform of T with respect to $S' \to S$. Therefore, by [22], see 2.19, we may assume in addition to *a)-c)*, *e)* and *g)* that we have

h) Both X and T, defined as above, are flat over S.

This condition is also stable under further modifications of S. Thus we may normalize S and assume that

i) The scheme S is normal.

4.19. Assume we have X, S as above, satisfying *a)-c)*, *e)* and *g)-i)*. We will show that β extends to a morphism.

Take a point $s \in \mathrm{S}$. Denote by X_s, $\mathscr{C}_s$, T_s the fibres of X, $\mathscr{C}$, T over s. Note that T_s has pure dimension 1 as $\mathrm{T} \to \mathrm{S}$ is flat. We decompose these into irreducible components

$$\mathrm{X}_s = \mathrm{X}_1 \cup \ldots \cup \mathrm{X}_r,$$

$$\mathscr{C}_s = \mathrm{C}_1 \cup \ldots \cup \mathrm{C}_c,$$

$$\mathrm{T}_s = \mathrm{T}_1 \cup \ldots \cup \mathrm{T}_t.$$

We have the morphisms $\mathrm{pr}_1 : \mathrm{T} \to \mathscr{C}$ and $\mathrm{pr}_2 : \mathrm{T} \to \mathrm{X}$. We remark that these are birational, i.e. modifications.

4.20. *Lemma. — In the situation above.*

(i) *For each* i, $1 \leqslant i \leqslant r$ *there is exactly one* $j = j_i$ *such that* $\mathrm{pr}_2(\mathrm{T}_j) = \mathrm{X}_i$. *There is an open subscheme* $\mathrm{V} = \mathrm{V}_i \subset \mathrm{X}$ *such that* $\mathrm{V} \cap \mathrm{X}_i$ *is nonempty, and* $\mathrm{pr}_2^{-1}(\mathrm{V}) \to \mathrm{V}$ *is an isomorphism. The morphism* $\mathrm{pr}_1 : \mathrm{T}_j = \mathrm{T}_{j_i} \to \mathscr{C}_s$ *is nonconstant.*

(ii) *For each* i, $1 \leqslant i \leqslant c$ *there is exactly one* $j = j_i$ *such that* $\mathrm{pr}_1(\mathrm{T}_j) = \mathrm{C}_i$. *There is an open subscheme* $\mathrm{V} = \mathrm{V}_i \subset \mathscr{C}$ *such that* $\mathrm{V} \cap \mathrm{C}_i$ *is nonempty, and* $\mathrm{pr}_1^{-1}(\mathrm{V}) \to \mathrm{V}$ *is an isomorphism.*

Proof. — Let $\mathrm{W} \subset \mathrm{X}_s$ be the finite set of closed points where the morphism $\mathrm{pr}_2 : \mathrm{T}_s \to \mathrm{X}_s$ has one-dimensional fibres. Take a closed point $x \in \mathrm{X}_s$, $x \notin \mathrm{W}$. The proper morphism pr_2 has a finite fibre at x, hence there exists an open neighbourhood $\mathrm{V} \subset \mathrm{X}$ of x such that $\mathrm{pr}_2^{-1}(\mathrm{V}) \to \mathrm{V}$ is finite, 2.7. The morphism $\mathrm{T} \to \mathrm{X}$ is birational, hence $\mathrm{pr}_2^{-1}(\mathrm{V}) \to \mathrm{V}$ is a finite modification. If x lies in the smooth locus of $f : \mathrm{X} \to \mathrm{S}$, and this excludes only finitely many closed points of X_s by *b)*, then x is a normal point of X, as S is normal. In this case we may assume that V is normal. Thus the finite birational morphism $\mathrm{pr}_2^{-1}(\mathrm{V}) \to \mathrm{V}$ is an isomorphism. Note that for any irreducible component $\mathrm{X}_i \subset \mathrm{X}_s$, we may choose such an $x \in \mathrm{X}_i$. The first two statements of (i) follow.

The same argument applied to $\mathrm{pr}_1 : \mathrm{T} \to \mathscr{C}$ proves (ii).

Let $\mathrm{T} \to \mathrm{X}' \to \mathrm{X}$ be the Stein factorization of $\mathrm{pr}_2 : \mathrm{T} \to \mathrm{X}$. Then $\mathrm{X}' \to \mathrm{X}$ is a finite modification, hence an isomorphism over the locus of points where X is normal. We conclude that $\mathrm{pr}_2^{-1}(x)$ is connected, for any normal point x of X, in particular for $x \in \mathrm{sm}(\mathrm{X}/\mathrm{S})$.

Choose i, $1 \leqslant i \leqslant r$, and let $j = j_i$ as in (i). Let $\{\alpha, \beta, \gamma\} \subset \{1, \ldots, n\}$ be such that $x_\alpha = \sigma_\alpha(s)$, $x_\beta = \sigma_\beta(s)$ and $x_\gamma = \sigma_\gamma(s)$ are distinct, lie on X_i and in $\mathrm{sm}(\mathrm{X}/\mathrm{S})$. Such a triple exists by *e)*. Suppose that $\mathrm{pr}_1 : \mathrm{T}_j \to \mathscr{C}_s$ is constant and let $c \in \mathscr{C}_s$ be the unique point in its image. Let $T_\alpha = pr_2^{-1}(x_\alpha)$, $T_\beta = pr_2^{-1}(x_\beta)$ and $T_\gamma = pr_2^{-1}(x_\gamma)$; these are connected schemes by the remark above. Note that $t_\alpha = (\tau_\alpha(s), \sigma_\alpha(s)) \in Z$ is a point of T_α, and similarly for t_β and t_γ. Further, $T_\alpha \cap T_j \neq \emptyset$, since the complete curve T_j dominates X_i and $x_\alpha \in X_i$. Again we have a similar statement for β and γ. Finally, putting $c_\alpha = pr_1(t_\alpha) = \tau_\alpha(s)$, etc., we have $\#\{c_\alpha, c_\beta, c_\gamma\} = 3$, as $(\mathscr{C}_s, \tau_1(s), \ldots, \tau_n(s))$ is a stable n-pointed curve.

There are two cases; both leading to an absurdity.

Case 1. — Suppose $c \notin \{c_\alpha, c_\beta, c_\gamma\}$. This means that $pr_1(T_\alpha)$ has dimension 1 as it connects $c = pr_1(T_\alpha \cap T_j)$ with the point c_α; similarly for $pr_1(T_\beta)$ and $pr_1(T_\gamma)$. By (ii) the curves $pr_1(T_\alpha)$, $pr_1(T_\beta)$ and $pr_1(T_\gamma)$ have no component in common. Thus we get at least three distinct components through the point $c \in \mathscr{C}_s$. This contradicts the semi-stability of the curve $\mathscr{C}_s$.

Case 2. — Suppose now that $c \in \{c_\alpha, c_\beta, c_\gamma\}$, say $c = c_\alpha$. In this case we see that the curves $pr_1(T_\beta)$ and $pr_1(T_\gamma)$ meet in the labeled point $c = c_\alpha$. This contradicts the stability of $(\mathscr{C}_s, \tau_1(s), \ldots, \tau_n(s))$.

4.21. In the situation of 4.19, the lemma implies that the morphism $pr_1 : T \to \mathscr{C}$ has finite fibres, hence is a finite morphism. We remark that $\mathscr{C}$ is a normal scheme: it is flat over a normal excellent scheme, with reduced fibres of dimension 1, hence condition S_2 is fullfilled; the smooth locus of $\mathscr{C} \to S$ is dense in all fibres and the generic fibre is smooth, hence $\mathscr{C}$ is regular in codimension 1; apply the criterium of Serre ($S_2 + R_1 \Rightarrow$ normal). Thus the birational finite morphism $pr_1 : T \to \mathscr{C}$ is an isomorphism.

We conclude that the properties *a)-c)*, *e)* and *g)* on data $X \to S$, σ_i as in 4.18 imply that the rational map β extends to a birational morphism $\beta : \mathscr{C} \to X$, at least after replacing S by a modification and $\mathscr{C}$ and X by their strict transforms.

4.22. We continue the discussion of the proof of Theorem 4.1, from the point we left it in the beginning of 4.18. Thus we have a pair (X, Z) satisfying (i)-(iv), (vi) *a)-g)*. We apply the results of 4.18-4.21 and find a modification $\psi : Y' \to Y$, such that β' extends. Once again using 4.15 we may replace Y by Y', etc., and assume that β extends to $\beta : \mathscr{C} \to X$ and we still have (i)-(iv), (vi) *a)-g)*. Consider the closed subset $\beta^{-1}(Z)$, which is pure of codimension 1 in $\mathscr{C}$ by (iv). Let E' be an irreducible component of $\beta^{-1}(Z)$. If we do not have $E' = \tau_i(Y)$ for some i, then the image D' of E' in Y has codimension 1 in Y, as β is an isomorphism over the open set U of (vi) *g)*. Thus there is a closed subset $D \subset Y$ such that we have $\beta^{-1}(Z) \subset \tau_1(Y) \cup \ldots \cup \tau_n(Y) \cup f^{-1}(D)$ and such that $\mathscr{C} \to Y$ is smooth over $Y \setminus D$.

We replace X by $\mathscr{C}$ and Z by $\tau_1(Y) \cup \ldots \cup \tau_n(Y) \cup f^{-1}(D)$, see 4.4 and 4.9. At this point we apply the induction hypothesis: there exists a nonsingular projective variety Y' and a generically étale alteration $\psi : Y' \to Y$ such that the closed subset $\psi^{-1}(D)$ is a strict normal crossings divisor on Y. Pulling back the family $\mathscr{C}$ to a family $\mathscr{C}'$

over Y′ and applying 4.4 once again, we reduce to the situation described in 4.23 below. (We drop the stability hypothesis, since we will not need it any more.)

4.23. *Situation.* — Here Y is a nonsingular projective variety over the algebraically closed field k, $D \subset Y$ is a divisor with strict normal crossings, and $f: X \to Y$ is a semi-stable curve, smooth over $Y \setminus D$. The closed subset $Z \subset X$ equals $\tau_1(Y) \cup \ldots \cup \tau_n(Y) \cup f^{-1}(D)$, where τ_i, $1 \leqslant i \leqslant n$ are mutually disjoint sections into the smooth locus of f, i.e. $\tau_i : Y \to \mathrm{sm}(X/Y)$.

4.24. Using the modification of Lemma 3.2 we reduce to the situation 4.23, where we have in addition that $\operatorname{codim}(\operatorname{Sing}(X), X) \geqslant 3$. (Of course the sections τ_i still map into the smooth locus of f; in fact, Z is already everywhere a divisor with normal crossings, except in the singular points of X. Furthermore, it is a divisor, as D is a divisor and $\tau_i(Y)$ is a divisor.) The situation is further explained in 3.5. Using these explanations we see that we reduce to the situation described in 4.25 below. (Note that there we consider only closed points, so that the situation is automatically split.)

4.25. *Situation.* — Here X is a projective variety of dimension d over an algebraically closed field k. We have a divisor $Z \subset X$. Let $x \in X(k)$ be an arbitrary closed point; either of the following two conditions holds:

(i) x is a nonsingular point of X. In this case Z is a normal crossings divisor at x (or $x \notin Z$).

(ii) x is a singular point of X. In this case there are integers $2 \leqslant s \leqslant r \leqslant d - 1$ such that the completion of the local ring $\mathcal{O}_{X,x}$ is isomorphic to

$$k[[u, v, t_1, \ldots, t_{d-1}]]/(uv - t_1 \cdot \ldots \cdot t_s)$$

and Z is defined by $t_1 \cdot \ldots \cdot t_r = 0$.

Finally, the components of the singular locus of X are nonsingular.

4.26. Assume (X, Z) as in 4.25. Let $E \subset X$ be an irreducible component of Sing(X). Let $\pi : X' \to X$ be the blowing up of X in the ideal sheaf of E, and put $Z' = \pi^{-1}(Z)_{\mathrm{red}}$.

4.27. *Claim.* — *The pair* (X′, Z′) *is as described in* 4.25. *The number of components of* Sing(X′) *is one less than the number of components of* Sing(X).

Again the proof is a nice exercise in blowing up: Since E is smooth, its ideal in the rings of (ii) is given by (u, v, t_1, t_2) after renumbering. Before we give some computations, let us describe the singular locus of X′. Let $E' \subset X$ be another irreducible component of the singular locus of X. Let $\widetilde{E}' \subset X'$ be the strict transform of E′. This equals the blowing up of E′ in the nonsingular closed subscheme $E' \cap E$ (scheme-theoretically), hence $\widetilde{E}'$ is nonsingular. Then Sing(X′) is the union of the $\widetilde{E}'$ so obtained.

We blow up the scheme

$$\operatorname{Spec} k[u, v, t_1, \ldots, t_{d-1}]/(uv - t_1 \cdot \ldots \cdot t_s)$$

in the ideal (u, v, t_1, t_2). We get four charts associated to the coordinates u, v, t_1, t_2. By symmetry, we need only deal with two of these.

Chart: " $u \neq 0$ ". — Here we have coordinates u, v, $t_1, \ldots, t_{d-1}, v', t'_1, t'_2$ and equations,

$$v = uv', \quad t_1 = ut'_1, \quad t_2 = ut'_2 \quad \text{and} \quad v' - t'_1 \cdot t'_2 \cdot t_3 \cdot \ldots \cdot t_s = 0.$$

Clearly, this is smooth and Z' is given by $ut'_1\, t'_2\, t_3 \cdot \ldots \cdot t_r = 0$, a normal crossings divisor.

Chart: " $t_1 \neq 0$ ". — Here we have coordinates $u, v, t_1, \ldots, t_{d-1}, u', v', t'_2$ and equations

$$u = t_1 u', \quad v = t_1 v', \quad t_2 = t_1 t'_2 \quad \text{and} \quad u' v' - t'_2 \cdot t_3 \cdot \ldots \cdot t_s = 0.$$

The divisor Z' is given by $t'_2 \cdot t_3 \cdot \ldots \cdot t_s \cdot t_1 = 0$. Clearly the singularities are of the type described in (ii). The irreducible component $u' = v' = t'_2 = t_3 = 0$ of the singular locus maps onto $u = v = t_2 = t_3 = 0$, the component $u' = v' = t_3 = t_4 = 0$ is the strict transform of the component $u = v = t_3 = t_4 = 0$.

4.28. By repeatedly blowing up (X, Z) as in 4.26 we finally get the situation that X is nonsingular and Z is a normal crossings divisor. It is well known that by blowing up X further we can reach the situation where Z has strict normal crossings, see 2.4. This finishes the proof of Theorem 4.1.

5. Alterations and curves

5.1. Let $f: X \to S$ be a proper morphism of Noetherian schemes. We define a number of conditions on f:

a) All fibres of f are nonempty and equidimensional of dimension 1.

b) The smooth locus of f is dense in all fibres of f.

c) f is flat.

Let $\sigma_1, \ldots, \sigma_n : S \to X$ be sections of f. We also consider the following conditions (here we usually assume *a)* and *b)*):

d) For any geometric point $\bar{s}$ of S and any singular point $\bar{x} \in X_{\bar{s}}$, there is an i such that $\bar{x} = \sigma_i(\bar{s})$.

e) For any geometric point $\bar{s}$ of S and any irreducible component C of $X_{\bar{s}}$, there exist $i, j, k \in \{1, \ldots, n\}$ such that $\sigma_i(\bar{s})$, $\sigma_j(\bar{s})$ and $\sigma_k(\bar{s})$ are three distinct points lying on $C \cap \operatorname{sm}(X/S)$.

5.2. *Lemma.* — *Let* S *be an excellent integral scheme. Let* $f: X \to S$ *be a projective morphism satisfying* 5.1 *a) and b). There exists a projective alteration* $\psi: S' \to S$, *and sections* $\sigma_1, \ldots, \sigma_n : S' \to X \times_S S'$ *such that property* 5.1 *e) is satisfied.*

Proof. — The assertion is local on S in the following sense. Suppose that $S = \bigcup U_\alpha$ is a finite covering of S by open affines, $\psi_\alpha : U'_\alpha \to U_\alpha$ are projective alterations, and $\sigma_1^\alpha, \ldots, \sigma_{n_\alpha}^\alpha$ are sections of $X \times_{U_\alpha} U'_\alpha \to U'_\alpha$ such that *e)* holds for geometric points of U'_α. Then we can construct $S' \to S$, $\sigma_1, \ldots, \sigma_n$ as in the lemma.

We can choose alterations $S'_\alpha \to S$, dominating U'_α over U_α such that the sections σ_i^α extend so S'_α, see Lemma 5.5. Next, we choose an alteration $S' \to S$ which dominates all S_α, see 5.4. As sections $\sigma_1, \ldots, \sigma_n$ we take the pullbacks of all the sections σ_i^α. It is easy to verify that $S', \sigma_1, \ldots, \sigma_n$ satisfies *e)*.

Let $s \in S$ be a closed point. We will find an affine open neighbourhood U of s in S, such that we can solve the problem over U. By the above and since S is Noetherian, this will suffice.

First we take U affine such that X is projective over U. Let $\mathscr{L}$ be very ample on X over U. Let $n \geqslant 3$ be so large that $f_* \mathscr{L}^{\otimes n} \to H^0(X_s, \mathscr{L}^{\otimes n}|_{X_s})$ is surjective. (For example if H^1 is zero; this remains true after flat base change). There exists a finite separable extension $k \subset k'$ and a section $t \in \Gamma(X \otimes k', (\mathscr{L} \otimes k')^{\otimes n})$, such that the divisor $H(t) \subset X \otimes k'$ defined by t is (i) a scheme finite étale over Spec k' and (ii) $H(t) \subset sm(X/S)$. (Bertini Theorem on the generically smooth curve X_s over s; actually (ii) is implied by (i).) There exists a $\psi : U' \to U$ finite étale such that $\psi^{-1}(s) = s'$ and $\kappa(s') \cong k'$, see Lemma 5.7, possibly we have to shrink U. We can lift the section t to a section $\tilde{t} \in \Gamma(X_{U'}, \mathscr{L}^{\otimes n})$. Let $H(\tilde{t})$ be the divisor it defines. The locus in U', where $H(\tilde{t}) \to U'$ has fibres of dimension 1 is closed and avoids s'. Removing its image in U from U, we may assume that $H(\tilde{t})$ is finite over U'. The complement of $sm(X/S)_{U'}$ in $H(\tilde{t})$ is closed and disjoint from $H(\tilde{t}) \cap X_{s'}$. Thus we may remove its image in U from U (by finiteness this is closed), and we see that $H(\tilde{t}) \subset sm(X_{U'}/U')$. It follows that $H(\tilde{t}) \to U'$ is flat, as it is a relative divisor in a scheme smooth over U'. In particular, $H(\tilde{t}) \to U'$ is étale at all points of $H(\tilde{t}) \cap X_{s'} = H(t)$, by our choice of t. We remove from U the image of the closed subset of points of $H(\tilde{t})$, where $H(\tilde{t}) \to U'$ is not étale. Thus, finally, we have that $H(\tilde{t}) \to U'$ is finite étale.

Let $\bar{u}'$ be a geometric point of U', and let C be an irreducible component of $X_{\bar{u}'}$. Since $\mathscr{L}|_C$ has degree at least 1, we see that $\mathscr{L}^{\otimes n}|_C$ has degree at least three. Since $H(\tilde{t}) \to U'$ is finite étale, we see that $H(\tilde{t}) \cap C$ consists of at least three distinct points, contained in $sm(X_{U'}/U')$ by the above. We apply Lemma 5.6 to the triple $(U', X_{U'}, H(\tilde{t}) \subset X_{U'})$. This gives $U'' \to U'$ and sections $\sigma_i : U'' \to X_{U''}$ such that *e)* holds for geometric points of U''. This finishes the proof. Q.E.D.

5.3. *Lemma. — Let* S *be an excellent integral scheme. Let* $f : X \to S$ *be a projective semi-stable curve. There exists a projective alteration* $\psi : S' \to S$, *and sections* $\sigma_1, \ldots, \sigma_n : S' \to X \times_S S'$ *such that property* 5.1 *d) is satisfied.*

Proof. — The same arguments as in the proof of the preceding lemma show that the question is local on S. We take a closed point $s \in S$ and find an affine open neighbourhood U of s over which the problem can be solved.

First we take U affine such that X is projective over U. Let $\mathscr{L}$ be very ample on X over U. This time we take n so large that

$$(*) \qquad f_* \mathscr{L}^{\otimes n} \to f_*(\mathscr{L}^{\otimes n}|_{\mathrm{Sing}(f)})$$

is surjective. Choose a section $t \in \Gamma(X_s, \mathscr{L}^{\otimes n}|_{X_s})$ such that $\mathrm{Sing}(f)_s \subset H(t)$ and no component of X_s is contained in $H(t)$. (Here we may have to enlarge n a bit.) The surjectivity above implies that we may choose a lift $\tilde{t} \in \Gamma(X_U, \mathscr{L}^{\otimes n})$ of t which lies in the kernel of $(*)$. Thus $\mathrm{Sing}(f)_U \subset H(t)$. Arguing as in the proof of Lemma 5.2, we may after shrinking U assume that $H(\tilde{t})$ is finite over U, hence it is flat over U. We apply Lemma 5.6 to the triple U, X_U, $H(\tilde{t})$, and conclude as before. (Note that since we have a semi-stable curve, the formation of singular locus commutes with base change.) Q.E.D.

5.4. *Lemma. — Let* $S'_1, \ldots, S'_n$ *be a finite set of alterations of an integral Noetherian scheme* S. *There exists a projective alteration* $S' \to S$ *which dominates all the* S'_i.

Proof. — Take S′ to be an arbitrary irreducible component of $S'_1 \times_S \ldots \times_S S'_n$ which dominates S. If it is not projective over S, use Chow's lemma to get it so. Q.E.D.

5.5. *Lemma. — Let* $f : X \to S$ *be a proper morphism, with* S *integral and Noetherian. Let* $U \subset S$ *be open,* $U' \to U$ *an alteration and* $\sigma_1, \ldots, \sigma_n : U' \to X_{U'}$ *sections of* X *over* U′. *There exist an alteration* $\psi : S' \to S$, *and an* S*-morphism of* $\psi^{-1}(U)$ *to* U′, *such that the sections* σ_i *extend to sections* $\sigma_i : S' \to X'_S$.

Proof. — Apply Chow's lemma to $U' \to U$, to see that we may assume there exists a closed immersion $i : U' \to \mathbf{P}^N_U$. Let S′ be the (schematic) closure in $\mathbf{P}^N \times (X \times_S \ldots \times_S X)$ of the image of

$$U' \xrightarrow{i, \sigma_j} \mathbf{P}^N \times (X \times_S \ldots \times_S X)_U.$$

Everything is clear. Q.E.D.

5.6. *Lemma. — In this lemma* S *is integral and excellent. Let* $f : X \to S$ *be a proper morphism, and let* $Z \subset X$ *be a closed subset, finite and flat over* S. *There exists a finite alteration* $S' \to S$, *and sections* $\sigma_1, \ldots, \sigma_n$ *such that* $\varphi^{-1}(Z) = \bigcup_i \sigma_i(S')$ (*set theoretically*), *where* $\varphi : X \times_S S' \to X$ *denotes the projection morphism.*

Proof. — Let $Z = \bigcup Z_i$ be the decomposition of Z into its irreducible components. The field extensions $R(S) \subset R(Z_i)$ are finite; choose a finite normal field extension $R(S) \subset L$, such that $R(Z_i) \subset L$ for all i. Let S′ be the normalization of S in L (as S is excellent, this is finite over S). We have the diagram:

$$\begin{array}{ccccc} Z \times_S S' & \longleftarrow & Z_i \times_S S' & \longrightarrow & Z_i \\ \downarrow & & \downarrow & & \downarrow \\ S' & \xleftarrow{\mathrm{id}} & S' & \longrightarrow & S. \end{array}$$

Since the left vertical arrow is finite and flat all components of $Z \times_S S'$ dominate S'. Thus we see that the irreducible components of $Z \times_S S'$ are the irreducible components of $Z_i \times_S S'$ which dominate S'. By our choice of L, all these irreducible components Z_{ij} of $Z_i \times_S S'$ are finite and birational over S'. Since S' is normal we get $Z_{ij} \cong S'$. Hence we get $\sigma_{ij} : S' \to Z_i \times_S S'$ such that $Z \times_S S' = \bigcup \sigma_{ij}(S')$. Q.E.D.

5.7. *Lemma.* — *Let* U *be an affine scheme,* $u \in U$ *a closed point and* $\kappa(u) \subset k'$ *a finite separable extension. There exists a finite free* $\psi : U' \to U$ *such that* $\psi^{-1}(u) = u'$ *with* $\kappa(u') \cong k'$ *as field extensions of* $\kappa(u)$. *There is an open neighbourhood of* u *in* U *over which* ψ *is étale.*

Proof. — Write $k' = \kappa(u)[x]/(f(x))$, with f monic. Lift f to a monic polynomial F in $\Gamma(U, \mathcal{O}_U)$, and use this do define U'. Q.E.D.

5.8. *Theorem.* — *Let* $f: X \to S$ *be a projective morphism of integral excellent schemes. Assume that*

a) all fibres of f *are nonempty and equidimensional of dimension 1;*
b) the smooth locus of f *is dense in all fibres of* f.

There exists a diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\varphi_1} & X \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} \\ S_1 & \xrightarrow{\psi_1} & S, \end{array}$$

where ψ_1 *and* φ_1 *are alterations and* f_1 *is a projective split semi-stable curve with smooth generic fibre.*

Suppose that $Z \subset X$ *is a proper closed subset. We may choose the diagram above such that there are mutually disjoints sections* $\sigma_1, \ldots, \sigma_n$ *of* S_1 *into* $\mathrm{sm}(X_1/S_1)$ *and a divisor* $D_1 \subset S_1$ *with the property that*

$$\varphi_1^{-1}(Z)_{\mathrm{red}} \subset f_1^{-1}(D_1)_{\mathrm{red}} \cup \sigma_1(S_1) \cup \ldots \cup \sigma_n(S_1).$$

5.9. The strategy of the proof is similar to the strategy used in the proof of Theorem 4.1. In particular, we will consider alterations $\psi : S' \to S$ and diagrams

$$\begin{array}{ccccc} X' & \longrightarrow & X \times_S S' & \xrightarrow{\mathrm{pr}_1} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{\mathrm{id}} & S' & \xrightarrow{\psi} & S. \end{array}$$

Here X' is an irreducible component of $X \times_S S'$ dominating X. (Such a component exists, as ψ is finite flat over a nonempty open part of S.) It follows that the generic fibre of f' is equidimensional of dimension 1, hence all fibres of f' have everywhere dimension at least 1 (2.7). Thus, since *a)* holds for pr_2, we get *a)* for f'. Property *b)* then follows for f' as it holds for pr_2. Denote $\varphi : X' \to X$ the obvious morphism; it is dominant and proper and finite over f^{-1} of the locus where ψ is finite. Hence it is an alteration. Note that $Z' = \varphi^{-1}(Z)_{\mathrm{red}}$ is a proper closed subset of X' also. It suffices to prove the theorem for the triple (X', Z', S').

5.10. Let $\eta \in S$ be the generic point of S. The curve X_η is irreducible, since X is irreducible. There exists a finite extension $\kappa(\eta) \subset L$ such that the normalization of $(X_\eta \otimes L)_{\mathrm{red}}$ is a union of smooth geometrically irreducible curves over Spec L. Let $\psi : S' \to S$ be the normalization of S in the field L; it is an alteration. Choose an irreducible component X' as in 5.9 and let $(X')^n \to X'$ be the normalization mapping. The scheme $\mathrm{sm}(X'/S')$ is normal as it is smooth over a normal scheme. Thus $(X')^n \to X'$ is an isomorphism outside a locus finite over S'. Obviously this implies that the conditions *a)* and *b)* hold for the morphism $(X')^n \to S'$. Hence replacing S by S', X by $(X')^n$ and Z by its inverse image in $(X')^n$ we may assume that we have *a)*, *b)* and

c) the generic fibre of f is smooth and geometrically irreducible.

This property is preserved by replacing X, S by X', S' as in 5.9; note that in this case X' is actually equal to the reduction of the irreducible scheme $X \times_S S'$, i.e. X' is the strict transform of X (see 2.18 and 2.20).

5.11. Assume *a)-c)*. Let $U \subset S$ be a nonempty open subset such that $Z_U \to U$ is finite flat, see 2.7. By 5.6 there exists an alteration $U' \to U$ and sections $\sigma_1, \ldots, \sigma_n$ over U' such that $Z_{U'} = \sigma_1(U') \cup \ldots \cup \sigma_n(U')$ (set-theoretically). Extend $U' \to U$ to an alteration $S' \to S$ such that the sections σ_i extend to sections $\sigma_i : S' \to X_{S'}$, see 5.5. Take $X' = (X \times_S S')_{\mathrm{red}}$ and $Z' \subset X'$ as before. Then it is clear that $Z' = Z'_v \cup \sigma_1(S') \cup \ldots \cup \sigma_n(S')$, where Z'_v is vertical, i.e. $f'(Z'_v) \neq S'$. We blow up $S'' \to S'$ in the (ideal sheaf of the) closed subscheme $f'(Z'_v)$. Thus we get a divisor $D'' \subset S''$ and sections σ''_i such that $Z'' \subset (f'')^{-1}(D'') \cup \sigma''_1(S'') \cup \ldots \cup \sigma''_r(S'')$. Replacing f, X, S, Z by the objects with two dashes, we reduce to a situation where we have *a)-c)* and the following property:

d) $Z \subset f^{-1}(D)_{\mathrm{red}} \cup \sigma_1(S) \cup \ldots \cup \sigma_n(S)$ for a divisor $D \subset S$ and sections $\sigma_i : S \to X$, $i = 1, \ldots, n$.

Again this is stable for the process described in 5.9.

5.12. We apply Lemma 5.2, this gives $S' \to S$ and a set of sections σ'_i of X' to S' such that property 5.1 *e)* is satisfied for $X' \to S'$. Applying 5.9 and adding the sections σ'_i to the sections we already produced in 5.11 we reduce to the case where we have *a)-d)* and 5.1 *e)*. (We write *e)* instead of 5.1 *e)* in the sequel.)

5.13. There is a nonempty open subscheme $U \subset S$ such that $(X_U, \sigma_1|_U, \ldots, \sigma_n|_U)$ is a smooth stable n-pointed curve of genus g, where $g = g(X_\eta)$. This gives a 1-morphism $U \to \overline{\mathcal{M}}_{g,n}$. Let $U' \to U$ be an irreducible component dominating U of the scheme $\overline{M} \times_{\overline{\mathcal{M}}_{g,n}} U$, where $\overline{M}$ is an in 2.24. Let S' be the closure of U' in the scheme $\overline{M} \times S$.

Remark that $S' \to S$ is a projective alteration as $\overline{M}$ is projective over Spec $\mathbf{Z}$. It follows that in addition to *a)-e)* we may assume that we have

g) there exist a stable n-pointed curve $(\mathscr{C}, \tau_1, \ldots, \tau_n)$ over S, a nonempty open subscheme $U \subset S$ and an isomorphism $\beta : \mathscr{C}_U \to X_U$ mapping the section $\tau_i|_U$ to the section $\sigma_i|_U$.

5.14. We apply the results of 4.18-4.21. Thus we may (after modifying S) assume that β extends to a morphism $\beta : \mathscr{C} \to X$ of schemes over S. Put $X' = \mathscr{C}$ and $Z' = \beta^{-1}(Z)$. In order to get *d)* for the new pair (X', Z') it may be necessary to blow up S' a bit, since β blows up outside X_U and so Z' may have some "new" vertical components. However, note that the sections $\sigma'_i = \tau_i : S \to X'$ no longer satisfy *e)*, since as mentioned just now, $\beta : X' \to X$ blows up and X'_s has in general more components than X_s. Still, we have certainly reduced to the case described in 5.15 below.

5.15. *Situation.* — Here $f : X \to S$ is a projective semistable curve, with smooth generic fibre. There are mutually disjoint sections $\sigma_1, \ldots, \sigma_n : S \to X$ into sm(X/S) and $Z = f^{-1}(D)_{\mathrm{red}} \cup \sigma_1(S) \cup \ldots \cup \sigma_n(S)$ for some divisor $D \subset S$.

5.16. We note that the assumptions of 5.15 are preserved by replacing S by an alteration S' of S. Thus we apply Lemma 5.3 and Lemma 5.2 and we get additional sections $\sigma_{n+1}, \ldots, \sigma_{n+m}$ such that 5.1 *d)* and *e)* hold for $(X, \sigma_1, \ldots, \sigma_{n+m})$.

Going through the arguments of 5.13 and 5.14 once again, we see that after replacing S by an alteration of S, we have $(\mathscr{C}, \tau_1, \ldots, \tau_{n+m})$ a stable $n+m$-pointed curve over S and

$$\beta : \mathscr{C} \to X$$

mapping τ_i to σ_i. We know that for any geometric point $\bar{s}$ of S, any singular point $\bar{x} \in X_{\bar{s}}$ is equal to $\sigma_i(\bar{s})$ for some i, in view of 5.1 *d)*. Since the genus of $\mathscr{C}_{\bar{s}}$ is equal to the genus of $X_{\bar{s}}$, we either have that $\beta_{\bar{s}}^{-1}(\bar{x})$ is a point, or a string of smooth $\mathbf{P}^1$'s. But $\tau_i(\bar{s}) \in \beta_{\bar{s}}^{-1}(\bar{x})$ lies in the regular locus of $\mathscr{C}_{\bar{s}}$, hence we see that $\beta_{\bar{s}}^{-1}(\bar{x})$ is not a point. Thus we see that all components of $\mathscr{C}_{\bar{s}}$ are smooth: the components contracted under β are smooth curves of genus zero, the components mapping onto components of $X_{\bar{s}}$ by the preceding arguments.

We replace X by $\mathscr{C}$ as before and we reduce to Situation 5.15 with the additional information that all irreducible components of all geometric fibres $X_{\bar{s}}$ are smooth.

5.17. Assume $f : X \to S$, σ_i, D are as in 5.15, and all components of all geometric fibres of f are smooth. At this point we are essentially through. Just apply Lemmata 5.3 and 5.2 once again, to get additional sections $\rho_1, \ldots, \rho_r$ such that 5.1 *d)* and *e)* are satisfied for the system $(X, \rho_1, \ldots, \rho_r)$ over S. But then all singular points of all fibres X_s are rational: they lie in the finite set of points $\rho_1(s), \ldots, \rho_r(s) \in X_s(s)$ in view of property 5.1 *d)*. Similarly, all components are defined over $\kappa(s)$, since each component of X_s has a rational point in view of 5.1 *e)*. These components were smooth to begin with.

Thus $f: X \to S$ is a split semi-stable curve, and $f: X \to S$, σ_i, D solves our problem. Theorem 5.8 is proven.

6. Semi-stable alterations

6.1. In this section we work over a trait S, see 2.12. We want to prove an analogue of 4.1 for S-varieties. First we describe what is the best possible result that one can obtain by our methods.

6.2. If (X, Z) is a pair consisting of an S-variety and a closed subset $Z \subset X$ (also considered as a reduced closed subscheme of X), then we can write $Z = Z_f \cup Z'$, where $Z_f \to S$ is flat and Z' is contained in $f^{-1}(s)$. In fact, in the sequel we will only consider pairs (X, Z) with $f^{-1}(\{s\}) \subset Z$, so that $Z = Z_f \cup f^{-1}(\{s\})$.

6.3. We say that (X, Z) is a *strict semi-stable pair* if the following conditions are satisfied:

a) X is strict semi-stable over S, see 2.16.

b) Z is a divisor with strict normal crossings on X.

c) Let $Z_f = \bigcup_{i \in I} Z_i$ be the decomposition of Z_f in its irreducible components. For each $J \subset I$, the scheme $Z_J = \bigcap_{j \in J} Z_j$ is a disjoint union of S-varieties which are strict semi-stable over S.

In particular, *c)* implies that the schemes Z_J are flat over S. This concept is of most use when X is proper or projective over S.

6.4. (Local description of strict semi-stable pairs.) Let (X, Z) be a strict semi-stable pair over S and let $x \in X_s$ be a point. Choose a uniformizer $\pi \in \mathcal{O}_S$. We write $Z = X_s \cup Z_f$, with Z_f flat over S and we write the irreducible components of Z as follows: $X_s = \bigcup_i X_i$ and $Z_f = \bigcup_j Z_j$. Assume that $x \in Z_1, \ldots, Z_m$, not in other components Z_j (m might be zero), and $x \in X_1, \ldots, X_n$, not in other components X_i. We claim that the complete local ring A of X at x can be described as

$$A \cong C[[t_1, \ldots, t_n, s_1, \ldots, s_m]]/(\pi - t_1 \cdot \ldots \cdot t_n),$$

with C a Noetherian complete local formally smooth R-algebra and Z_j (resp. X_i) given by $s_j = 0$ (resp. $t_i = 0$).

By 2.16, since X is semi-stable, we already have that

$$A \cong B[[t_1, \ldots, t_n]]/(\pi - t_1 \cdot \ldots \cdot t_n),$$

and X_i given by $t_i = 0$. Suppose $s_j \in A$, $j = 1, \ldots, m$ defines the component Z_j at x. By semi-stability the complete local ring $\overline{B}/(\bar{s}_1, \ldots, \bar{s}_m)$ is a formally smooth

$\kappa(s)$-algebra, and $\bar{s}_1, \ldots, \bar{s}_m$ form part of a regular system of parameters of $\bar{\mathrm{B}}$. Lifting $\bar{\mathrm{C}} = \bar{\mathrm{B}}/(\bar{s}_1, \ldots, \bar{s}_m)$ to a formally smooth R-algebra, and arguing as in 2.8, we derive the claim.

On the other hand, suppose that for all points $x \in \mathrm{X}_s$, we have that $\mathrm{A} \cong \mathrm{C}[[t_1, \ldots, t_n, s_1, \ldots, s_m]]/(\pi - t_1 \cdot \ldots \cdot t_n)$, with C a Noetherian complete local formally smooth R-algebra and that Z_j (resp. X_i) given by $s_j = 0$, $j = 1, \ldots, m$ (resp. $t_i = 0$, $i = 1, \ldots, n$) are the only components of Z passing through x. Then (X, Z) is a strict semi-stable pair.

6.5. *Theorem. — Let* X *be an* S*-variety and let* $\mathrm{Z} \subset \mathrm{X}$ *be a proper closed subset (with* $f^{-1}(\{s\}) \subset \mathrm{Z}$*). There exist a trait* S_1 *finite over* S, *an* S_1*-variety* X_1, *an alteration of schemes over* S

$$\varphi_1 : \mathrm{X}_1 \to \mathrm{X}$$

and an open immersion $j_1 : \mathrm{X}_1 \to \bar{\mathrm{X}}_1$ *of* S_1*-varieties, with the following properties*:

(i) $\bar{\mathrm{X}}_1$ *is projective* S_1*-variety with geometrically irreducible generic fibre, and*

(ii) *the pair* $(\bar{\mathrm{X}}_1, \varphi_1^{-1}(\mathrm{Z})_{\mathrm{red}} \cup \bar{\mathrm{X}}_1 \backslash j_1(\mathrm{X}_1))$ *is strict semi-stable*, 6.3.

6.6. *Diagram.*

$$\begin{array}{ccccc} \bar{\mathrm{X}}_1 & \xleftarrow{j_1} & \mathrm{X}_1 & \xrightarrow{\varphi_1} & \mathrm{X} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{S}_1 & \xrightarrow{\mathrm{id}} & \mathrm{S}_1 & \longrightarrow & \mathrm{S}. \end{array}$$

6.7. The strategy of the proof is the same as in the case of varieties, compare 4.4 and 4.9. Again we will argue by induction on the relative dimension d of X over S. The case $\dim \mathrm{X}/\mathrm{S} = 0$ is all right. (We note that the case $d = 1$ does not trivially follow from the stable reduction theorem.) We do not repeat all the arguments; we just remark that if $\varphi : \mathrm{X}' \to \mathrm{X}$ is a modification or an alteration, then X' is an S-variety too (as X'_η is nonempty). Thus arguing as in 4.6 and 4.7 we reduce to the case where

(i) X is projective over S.

As in 4.8 we see that we may assume (in addition to (i)):

(ii) There exists a divisor $\mathrm{D} \subset \mathrm{X}$, such that Z is the support of D.

6.8. A new feature is that we are going to perform base change with respect to S. Suppose $\mathrm{S}' \to \mathrm{S}$ is a finite morphism of traits. Put $\mathrm{X}_{\mathrm{S}'} = \mathrm{X} \times_{\mathrm{S}} \mathrm{S}'$ and let $\mathrm{X}' \subset \mathrm{X}_{\mathrm{S}'}$ be an irreducible component, considered as a reduced closed subscheme. The morphism $\varphi : \mathrm{X}' \to \mathrm{X}$ is finite and dominant (since $\mathrm{X}_{\mathrm{S}'} \to \mathrm{X}$ is finite and flat), hence an alteration. Also, X' is an S'-variety. Put $\mathrm{Z}' = \varphi^{-1}(\mathrm{Z})$. The result of the theorem for the pair (X', Z') over S' implies the result for the pair (X, Z) over S.

6.9. There exists a finite extension $\mathrm{R}(\mathrm{S}) \subset \mathrm{L}$ such that some quotient field K of the Artinian semi-local ring $\mathrm{R}(\mathrm{X}) \otimes_{\mathrm{R}(\mathrm{S})} \mathrm{L}$ is separable over L and L is algebraically closed in K. Let $S' \to S$ be the associated finite extension of traits, i.e. $R(S') \cong L$. Let $X' \subset X_{S'}$ be the irreducible component corresponding to K. Performing the base change as explained in 6.8, we see that we may assume

(iii) X_η is geometrically reduced and irreducible over $\kappa(\eta)$.

Note that the properties (i) and (ii) are preserved by the operation described in 6.8. Hence we may assume we have (i)-(iii). Note that, in this case, $X_{S'}$ is irreducible and reduced for all morphisms of traits $S' \to S$.

6.10. Assume (i)-(iii). Performing the normalization modification $\varphi : X' \to X$ gives (i)-(iii) and

(iv) X is a normal scheme.

6.11. Assume (i)-(iv). Let $\{\xi_1, \ldots, \xi_r\}$ be the generic points of the fibre X_s. Let $\mathcal{O}_j$ be the local ring of X at the point ξ_j, $1 \leqslant j \leqslant r$. Let $S' \to S$, given by $R \to R'$, be a finite morphism of traits. We want that, for each j, the normalization of $\mathcal{O}_j \otimes_R R'$ is formally smooth over R'. To get this, we apply 2.13 to the homomorphisms $R \to \mathcal{O}_j$. This gives $R \subset R_j$ for each j, and we take R' to be a finite common extension of all R_j. (See last statement of 2.13; we remark that we do not need to reduce as X_η is geometrically reduced.)

Let $\varphi : X' \to X_{S'}$ be the normalization morphism, and let $\xi'_1, \ldots, \xi'_{r'}$ be the generic points of $X'_{s'}$. By the above we see that the extensions $R' \subset \mathcal{O}_{X', \xi'_i}$ have $e = 1$ and separable residue field extensions. The morphism $X' \to S'$ is therefore smooth in a neighbourhood of the point ξ'_i. We replace X by X′ and Z by the inverse image of Z in X′ and S by S′. This gives that in addition to (i)-(iv) we may assume:

(v) $\mathrm{sm}(X/S)$ is dense in X_s.

We note that the properties (i)-(iii) and (v) are preserved by further base change $X \mapsto X_{S'}$ whereas (iv) in general is not. Of course, (iv) is preserved by base change with $S' \to S$ finite étale. (Actually, we will note use (iv) any further.)

6.12. Assume (i)-(v). We claim that there exists a finite étale extension $S' \to S$ of traits and a finite morphism

$$\pi : X_{S'} \to \mathbf{P}^d_{S'},$$

such that π is étale over $V \subset \mathbf{P}^d_{S'}$ open, with $V_{s'}$ nonempty.

Let $X \to \mathbf{P}^N_S$ be a closed immersion. We remark that the conditions of 2.11 are satisfied for the closed subscheme $X_s \subset \mathbf{P}^N_{\kappa(s)}$. We find a finite separable extension $\kappa(s) \subset k'$ and $p \in \mathbf{P}^N_{k'}$ such that $\mathrm{pr}_p : X_s \otimes k' \to \mathrm{pr}_p(X_s \otimes k')$ is birational. The conditions of 2.11 are satisfied for the closed subscheme $\mathrm{pr}_p(X_s \otimes k') \subset \mathbf{P}^{N-1}_{k'}$. We continue and we find a finite separable extension $\kappa(s) \subset k'$ and a linear subvariety $L \subset \mathbf{P}^N_{k'}$ of dimension $N - d - 1$ such that $L \cap X_s \otimes k' = \emptyset$ and

$$\mathrm{pr}_L : X_s \otimes k' \to \mathbf{P}^d_{k'}$$

is finite and étale over a nonempty open subset of $\mathbf{P}^d_{k'}$.

Let $S' \to S$ be the unique finite étale extension of traits such that $\kappa(s') \cong k'$ as $\kappa(s)$-extensions. Choose a linear subvariety $\tilde{L} \subset \mathbf{P}^N_{S'}$ lifting L, i.e. $\tilde{L} \otimes \kappa(s') = L$. Note that $\tilde{L} \cap X_{S'} = \emptyset$, as $\tilde{L} \cap X_{S'}$ is closed with empty special fibre. The morphism

$$\pi = \mathrm{pr}_L : X_{S'} \to \mathbf{P}^d_{S'}$$

satisfies the properties of the claim. Use 2.8.

6.13. Thus we reduce to the case where we have (i)-(v) and a finite morphism

$$\pi : X \to \mathbf{P}^d_S,$$

such that π is étale over $V \subset \mathbf{P}^d_S$ open, with V_s nonempty. There exists a point $p \in V(k')$ defined over a finite separable extension $\kappa(s) \subset k'$. We use base change 6.8 (with $S' \to S$ finite étale, $\kappa(s') \cong k'$) to get p defined over $\kappa(s)$. Take any lift of p to a section $\sigma : S \to \mathbf{P}^d_S$; note that $\sigma(S) \subset V$, as $\sigma(s) \in V$. As in the proof of 4.11, consider the space $\mathbf{P}^{d-1}_S$ of lines ℓ through σ and put

$$X' = \{(x, \ell) \in S \times_S \mathbf{P}^{d-1}_S \mid \pi(x) \in \ell\}.$$

In this case the morphism $\varphi : X' \to X$ blows up in the locus $\pi^{-1}(\sigma(S)) \subset \mathrm{sm}(X/S)$, which is finite étale over S. (In particular, X′ is an X-variety.)

Arguing exactly as in 4.11 we see that

$$f = \mathrm{pr}_2 : X' \to \mathbf{P}^{d-1}_S$$

has the following properties: *a)* All fibres of f are equidimensional of dimension 1 and nonempty. *b)* The smooth locus of f is dense in all fibres of f. If we replace X by X′ and Z by $\varphi^{-1}(Z)$, then we see that we have (i)-(v) and

(vi) There exists a morphism $f: X \to Y$ of projective S-varieties having the following two properties:
(vi) *a)* All fibres of f are nonempty and equidimensional of dimension 1.
(vi) *b)* The smooth locus of f is dense in all fibres of f.

6.14. We apply Theorem 5.8 to the morphism $f: X \to Y$ of (vi). Thus we get alterations φ, ψ fitting into the diagram

$$\begin{array}{ccc} X' & \xrightarrow{\varphi} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\psi} & Y \end{array}$$

where f' is a projective split semi-stable curve. There are mutually disjoint sections

$$\sigma_1, \ldots, \sigma_n : Y' \to \mathrm{sm}(X'/Y')$$

such that $\varphi^{-1}(Z) \subset (f')^{-1}(D') \cup \sigma_1(Y') \cup \ldots \cup \sigma_n(Y')$, for a divisor $D' \subset Y'$. Clearly, Y′ is also an S-variety. Let us enlarge D′ so that it contains $f'(\mathrm{Sing}(f'))_{\mathrm{red}}$.

We may apply our induction hypothesis to the pair (Y', D') over S. Thus we find a finite extension $S' \to S$, a projective strict semi-stable pair (Y'', D'') over S' and an alteration $\psi' : Y'' \to Y'$ of S-schemes such that $(\psi')^{-1}(D')_{\mathrm{red}} = D''$. The split semi-stable curve $f' : X' \to Y'$ pulls back to a split semi-stable curve $f'' : X'' \to Y''$, the mutually disjoint sections σ_i pull back to mutually disjoint sections σ_i'' into $\mathrm{sm}(X''/Y'')$ and we have $Z'' \subset (f'')^{-1}(D'') \cup \sigma_1''(Y'') \cup \ldots \cup \sigma_n''(Y'')$. Furthermore, we have that X'' is smooth over $Y''\backslash D''$.

Therefore we have reduced the problem to the situation described in 6.15. (Recall that we may enlarge Z, compare 4.9.)

6.15. *Situation.* — Here (Y, D) is a projective strict semi-stable pair over S, $f : X \to Y$ is a projective split semi-stable curve over Y, smooth over $Y\backslash D$. There are mutually disjoint sections $\sigma_1, \ldots, \sigma_n : Y \to X$ into the smooth locus of X over Y, such that

$$Z = (f)^{-1}(D) \cup \sigma_1(Y) \cup \ldots \cup \sigma_n(Y).$$

Finally, Y has geometrically irreducible generic fibre.

6.16. Assume we are in Situation 6.15. Consider the modification $\varphi : X' \to X$ described in Proposition 3.6. Since it has center in the singular locus of f, the sections σ_i lift to mutually disjoint sections σ_i' into $\mathrm{sm}(X'/Y)$. Let Z' be the (set-theoretic) inverse image of Z in X'. Clearly, we also have $Z' = (f')^{-1}(D) \cup \sigma_1'(Y) \cup \ldots \cup \sigma_n'(Y)$.

Thus we reduce to the case described in 6.15 with the additional information that X is a regular scheme. We claim that (X, Z) is a strict semi-stable pair over S in this case. If we show this, then the proof of Theorem 6.5 is finished: a projective curve over a projective scheme over S is projective over S, the assertion on geometric irreducibility follows from the corresponding property of Y.

To prove the claim we write $D = \bigcup_{i \in I} D_i$, decomposition into irreducible components. Let us write $f^{-1}(D_i) = \bigcup_{j \in J_i} Z_{ij}$ for the irreducible components of $f^{-1}(D_i)$. Take a point $x \in X$, with image y in Y. We have to show that the situation at x looks like the description given in 6.4, see the remark at the end of 6.4.

In case $x \in \mathrm{sm}(X/Y)$ and $x \notin \sigma_i(Y)$ for all i, this follows trivially from the fact that (Y, D) is strict semi-stable and the description given in 6.4 of the complete local ring A at y. (Just add one formal variable to the ring C.) If $x \in \mathrm{sm}(X/Y)$ and $x \in \sigma_i(Y)$ for some i, then we just have to add one more s-variable to the ring A.

The case $x \in X$ is a singular point of f. Let A be the complete local ring of Y at y, let B be the complete local ring of X at x. The situation looks as follows:

$$A \cong C[[t_1, \ldots, t_n, s_1, \ldots, s_m]]/(t_1 \cdot \ldots \cdot t_n - \pi),$$

the components of D passing through y are given as the zero sets of the elements $t_1, \ldots, t_n, s_1, \ldots, s_m$. We know, since X is regular and split, see proof of 3.6, that $B \cong A[[u, v]]/(uv - h)$, with $h = s_1$ or $h = t_1$ (up to a renumbering). In either case consider the associated component D_i of D, i.e. the one given by the equation $h = 0$ in A. There must be two Z_{ij} passing through x. (The following argument also occurs in the proof of 3.6.) Indeed, the image of $\chi : \operatorname{Spec} B/(u, v) \to X$ lies in $\operatorname{Sing}(f)$ and dominates D_i, hence by the assumption that f is split we get $Z_{i1}, Z_{i2} \subset f^{-1}(D_i)$ with $\operatorname{Im}(\chi) \subset Z_{i1} \cap Z_{i2}$, hence also $x \in Z_{i1} \cap Z_{i2}$. Therefore in both cases, the ideals $(u) \subset B$ and $(v) \subset B$ correspond to the traces of distinct components of Z on Spec B. Since this was already clear for the other ideals (t_i) and (s_j) (by the corresponding property for A), we see that B is as in 6.4. (In case $h = s_1$ we get one more parameter s; in case $h = t_1$ we get one more parameter t.) This ends the proof of Theorem 6.5.

7. Group actions and alterations

7.1. Let S be a Noetherian scheme, let D be a divisor on S and let G be a finite group acting on S. Assume that G preserves D, i.e. for all $g \in G$ we have $g(D) \subset D$. We say that D is a *G-strict normal crossings divisor on* S if (i) $D = \bigcup_{i \in I} D_i$ is a strict normal crossings divisor on S, and (ii) the orbit of a component D_i is a disjoint union of components of D. This means that $D_i \cap g(D_i) \neq \emptyset \Rightarrow D_i = g(D_i)$. It also gives $Z \cap g(Z) \neq \emptyset \Rightarrow g(Z) = Z$, if Z is an irreducible component of $D_i \cap D_{i'}$.

7.2. Let S be an excellent scheme and assume that D is a divisor with normal crossings on S. Let us write D^{norm} for the normalization of D, and write $n : D^{\mathrm{norm}} \to D$ for the normalization morphism. We define

$$D^{(i)} = \{ s \in D \mid \mathrm{rk}_{\kappa(s)}\, n^{-1}(s) = i \}.$$

These are locally closed subschemes of D, and they are regular schemes. (These assertions and various other assertions may be checked étale locally on S, in which case one can compute explicitely using a local equation $t_1 \cdot \ldots \cdot t_r = 0$ for D.) We have

$$D = \coprod_{i=1}^{i=\dim S} D^{(i)},$$

with $\dim D^{(i)} = \dim S - i$.

Assume that S has pure dimension d. We define a sequence of blowings up

$$\widetilde{S} = S^{(1)} \to S^{(2)} \to \ldots \to S^{(d-1)} \to S^{(d)} = S.$$

The map $S^{(d-1)} \to S^{(d)}$ is the blowing up in the closed subscheme $D^{(d)}$ of S. Thus we can view $D^{(i)}$ for $i < d$ as a locally closed subscheme of $S^{(d-1)}$. The map $S^{(d-2)} \to S^{(d-1)}$ is defined as the blowing up of the closure of $D^{(d-1)}$ in $S^{(d-1)}$; note that this equals the normalization of the closure of $D^{(d-1)}$ in S. Thus we can view $D^{(i)}$ for $i < d - 1$ as a locally closed subscheme of $S^{(d-2)}$. In general, $S^{(i-1)} \to S^{(i)}$ is defined as the blowing up of the closure of $D^{(i)}$ in $S^{(i)}$.

Let $\varphi : \widetilde{S} \to S$ be the composition of the morphisms above. It has the following properties: (i) The reduced inverse image $\widetilde{D} = \varphi^{-1}(D)_{\mathrm{red}}$ is a divisor with strict normal crossings. (ii) The set of components of $\widetilde{D}$ is naturally bijective to the union of the sets of components of the $D^{(i)}$, $i = 1, \ldots, d$. If $Z \subset D^{(i)}$ is a component, then it corresponds to the closure of $\varphi^{-1}(Z)$. (iii) Let us write $D^{(i)} = \amalg D_j^{(i)}$. The irreducible components corresponding to $D_j^{(i)}$ and $D_{j'}^{(i')}$ intersect if and only if $D_j^{(i)}$ is contained in the closure of $D_{j'}^{(i')}$ or vice versa.

These assertions may be proven by induction on dim S, using that $\widetilde{S} \to S$ induces a similar morphism $\widetilde{D}_i \to D_i$ for any component D_i of D.

Suppose in the situation above that the finite group G acts on the scheme S, preserving the divisor D. Since $\widetilde{S} \to S$ is defined intrinsically in terms of the pair (S, D), the action of G lifts to an action of G on $\widetilde{S}$. Moreover the divisor $\widetilde{D}$ is a G-strict normal crossings divisor on $\widetilde{S}$. This is clear as all the components in one G-orbit on the set of components of $\widetilde{D}$ are of the same type, i.e. they correspond to components of $D^{(i)}$ for a fixed i. By the above these do not intersect.

7.3. *Theorem. — Let k be an algebraically closed field, let* X *be a variety over k,* $G \subset \operatorname{Aut} X$ *a finite subgroup of the automorphism group of* X *over k, and* $Z \subset X$ *a* G*-stable proper closed subset. There exist an alteration*

$$\varphi_1 : X_1 \to X,$$

an open immersion $j_1 : X_1 \to \overline{X}_1$, *and a finite subgroup* $G_1 \subset \operatorname{Aut} \overline{X}_1$ *such that the following properties hold:*

(i) *The action of* G_1 *preserves the open subscheme* X_1, *there is a surjection* $G_1 \to G$ *such that* φ_1 *is equivariant for the induced actions of* G_1 *an* X_1 *and* X. *The field extension* $k(X)^G \subset k(X_1)^{G_1}$ *is purely inseparable.*

(ii) $\overline{X}_1$ *is a projective nonsingular variety over k.*

(iii) *The closed subset* $j_1(\varphi_1^{-1}(Z)) \cup \overline{X}_1 \setminus j_1(X_1)$ *is a* G_1*-strict normal crossings divisor in* $\overline{X}_1$, *see* 7.1.

7.4. *Corollary. — Let* X *be a variety over a field k. There exist a finite field extension* $k \subset k'$, *a component* X' *of* $X \otimes k'$, *a radicial morphism* $X'' \to X'$ *of varieties over* k' *and a modification* $Y \to X''$ *such that* Y *has only quotient singularities.*

7.5. The strategy of the proof of 7.3 is the same as in the proof of Theorem 4.1. We just point out those places where we have to use substantially different arguments.

7.6. Let $\varphi : X' \to X$ be a modification with X' quasi-projective over k. Write X'_g for the scheme X' considered as a scheme over X with structural morphism $X' \xrightarrow{\varphi} X \xrightarrow{g} X$. Let $G = \{g_1, \ldots, g_n\}$ and let

$$X'' \subset X'_{g_1} \times_X X'_{g_2} \times_X \ldots \times_X X'_{g_n}$$

be the irreducible component dominating X. Then X'' is quasi-projective, a modification of X and G acts on X''. Hence we may assume that X is quasi-projective.

7.7. By arguments similar to 7.6 and 4.7 we may assume that X is projective.

7.8. We blow up in the ideal sheaf of $Z \subset X$. The group G acts on the result. Hence we may assume that Z is the support of a divisor on X.

7.9. We may assume that X is normal, compare 4.10.

7.10. Assume that the pair (X, Z) of the theorem satisfies conditions (iii), (iv) and (v) of Section 4. Consider the quotient morphism $X \to X/G$; of course X/G is a normal projective variety. Let $B \subset X/G$ be the branch locus of the morphism $X \to X/G$. We apply 4.11 to the variety X/G and the closed subset $B \cup Z/G$. We get a diagram

$$\begin{array}{ccccc} X' & \longrightarrow & (X/G)' & \longrightarrow & \mathbf{P}^{d-1} \\ \downarrow & & \downarrow & & \\ X & \longrightarrow & (X/G) & & \end{array}$$

where X' is the fibre product. The morphism $f' : X' \to \mathbf{P}^{d-1}$ is G-equivariant (with the trivial action on $\mathbf{P}^{d-1}$) and has the following properties:

a) All fibres are nonempty, geometrically connected and equidimensional of dimension 1.

b) The smooth locus of f' is dense in all fibres of f'.

d) The morphism $f'|_{Z'} : Z' \to \mathbf{P}^{d-1}$ is finite.

We are not able to conclude that $f'|_{Z'}$ is generically étale, since the map of Z' to its image in X/G might be inseparable on some components. If we choose the fibration general enough, we may assume that X'_η is geometrically irreducible also. To see this we remark that, by a Bertini theorem, for $H \subset \mathbf{P}^N$ general, $\dim H = N - d + 1$ (as at the end of the proof of 4.11 applied to $X/G \hookrightarrow \mathbf{P}^N$), the inverse image of $X/G \cap H$ in X is irreducible. Thus we get

c)' X'_η is geometrically irreducible and $X'_\eta/G = (X/G)'_\eta$ is smooth over η.

7.11. Let us apply Lemma 4.13 to the situation above. This gives a relative divisor $H \subset X$ with properties (i), (ii) of Lemma 4.13. Of course we replace Z by $Z \cup \bigcup_{g \in G} g(H)$. This reduces us to a situation where we also have:

e) For all geometric points $\bar{y}$ of Y and any irreducible component C of $X_{\bar{y}}$,

$$\# \mathrm{sm}(Y) \cup C \cup Z \geqslant 3.$$

7.12. Thus we have reduced to the case where there is a "nice" G-equivariant fibration $X \to Y$ of projective varieties satisfying *a)-e)* above. In the following we are going to allow situations where G does not act trivially on Y. Thus we assume that G also acts on Y and that f is G-equivariant (with an obvious modification of property *c)'* above).

Suppose we are given a finite field extension $k(\mathrm{Y}) \subset \mathrm{L}$. Let $\mathrm{L} \subset \mathrm{L}'$ be a finite extension such that the field extension $k(\mathrm{Y})^{\mathrm{G}} \subset \mathrm{L}'$ is normal. (Such exist.) Write $\mathrm{H} = \mathrm{Gal}(\mathrm{L}'/k(\mathrm{Y})^{\mathrm{G}})$ and let Y' be the normal closure of Y in L'. Write

$$\mathrm{G}' = \mathrm{H} \times_{\mathrm{Gal}(k(\mathrm{Y})/k(\mathrm{Y})^{\mathrm{G}})} \mathrm{G}.$$

The map $\mathrm{G}' \to \mathrm{G}$ is surjective, and $k(\mathrm{Y})^{\mathrm{G}} \subset \mathrm{L}'^{\mathrm{G}'}$ is purely inseparable. Further, G' acts on Y'. At this point we can do base change with $\mathrm{Y}' \to \mathrm{Y}$ as in 4.15. The result will be a G'-equivariant morphism $\mathrm{X}' \to \mathrm{Y}'$, where $\mathrm{G}' \subset \mathrm{Aut}\, \mathrm{X}'$. This proves that we may assume that the function field of the variety Y is sufficiently big.

7.13. Therefore, after extending $k(\mathrm{Y})$, we may assume that the generic fibre of f is smooth. (Here one has to normalize X' compare with the argument of 5.10.) Thus we get:

c) The generic fibre of f is smooth.

Also, by the procedure of 4.16 we may assume that:

f) There are sections $\sigma_1, \ldots, \sigma_n : \mathrm{Y} \to \mathrm{X}$ of f such that $\mathrm{Z} = \sigma_1(\mathrm{Y}) \cup \ldots \cup \sigma_n(\mathrm{Y})$.

7.14. Arguing as in 4.17, we see that there exists an alteration $\mathrm{Y}' \to \mathrm{Y}$ such that we have a stable curve $\mathscr{C}$ over Y', which agrees with $\mathrm{X}_{\mathrm{Y}'}$ over an open subscheme of Y'. By 7.12, we may assume that $\mathrm{Y}' \to \mathrm{Y}$ is birational, i.e., is a modification. Arguing as in 7.6, we can dominate Y' by a modification on which the group G' acts. Thus we may assume that we have:

g) There exist a stable n-pointed curve $(\mathscr{C}, \tau_1, \ldots, \tau_n)$ over Y, a nonempty open subscheme $\mathrm{U} \subset \mathrm{Y}$ and an isomorphism $\beta : \mathscr{C}_{\mathrm{U}} \to \mathrm{X}_{\mathrm{U}}$ mapping $\tau_i|_{\mathrm{U}}$ into the section $\sigma_i|_{\mathrm{U}}$.

7.15. We know that β extends to a morphism after a modification of Y, see 4.18-4.21. As above, we can dominate this by a G-equivariant modification. Thus we may assume that $\mathscr{C} \to \mathrm{X}$ exists. We replace X by $\mathscr{C}$ and Z by the inverse image of the degeneracy locus union the sections τ_i. (Note that the group G will act on $\mathscr{C}$ over Y.)

7.16. We are in the situation where we can apply induction to the variety Y with closed subset $\mathrm{D} \subset \mathrm{Y}$ and as group the image of G into Aut Y. We again do pullback. Hence we arrive at the situation of 4.23 where the finite group G acts on the situation and D is a G-strict normal crossings divisor on Y.

The blowing ups that occur in 4.24 can be done G-equivariantly, more precisely, when one blows up in a component as in 3.4, then one can blow up in the G-orbit of this component, which will be a disjoint union of components in view of the G-strictness of D. The same argument works for the procedure described in 4.26. (See end of 7.1 and end of 3.5.) Finally, apply the results of 7.2 to make the divisor Z into a G-strict normal crossings divisor on X. This ends the proof of Theorem 7.3.

8. Arithmetic case

8.1. In this section K is a global field, and $R \subset K$ is a Dedekind ring with fraction field K. We put $S = \operatorname{Spec} R$. If $K \subset K'$ is a finite extension of fields, we let $S' = \operatorname{Spec} R'$, where R' is the integral closure of R in K'. A morphism $S' \to S$ so obtained will be called a *finite extension of Dedekind schemes*.

An S-variety will be an integral scheme X over S with $X \to S$ separated of finite type and flat.

8.2. *Theorem. — Let* X *be an* S*-variety and* $Z \subset X$ *a proper closed subset. There exists a diagram*

$$\begin{array}{ccccc} \overline{X}_1 & \xleftarrow{j_1} & X_1 & \xrightarrow{\varphi_1} & X \\ \downarrow & & \downarrow & & \downarrow \\ S_1 & \xleftarrow{\mathrm{id}} & S_1 & \xrightarrow{\psi_1} & S \end{array}$$

where:

a) $\psi_1 : S_1 \to S$ *is a finite extension of Dedekind schemes,*

b) φ_1 *is an alteration,* j_1 *is an open immersion and* $\overline{X}_1$ *is projective over* S,

c) the scheme $\overline{X}_1$ *is regular and the closed subset* $j_1(\varphi_1^{-1}(Z)) \cup \overline{X}_1 \backslash j_1(X_1)$ *is a strict normal crossings divisor in* $\overline{X}_1$,

d) over some nonempty open $U_1 \subset S_1$ *the scheme* $\overline{X}_1$ *is smooth and* $j_1(\varphi_1^{-1}(Z)) \cup \overline{X}_1 \backslash j_1(X_1)$ *is a relative normal crossings divisor over* U_1, *and*

e) for points $s_1 \in S_1$, $s_1 \notin U_1$ *the pair* $(\overline{X}_1, j_1(\varphi_1^{-1}(Z)) \cup (\overline{X}_1)_{s_1} \cup \overline{X}_1 \backslash j_1(X_1))$ *is a strict semi-stable pair over the completion* S_{1/s_1} *of* S_1 *in* s_1, *see* 6.3.

8.3. We use the same techniques as those employed in Section 6. We may assume that $\dim X/S \geqslant 1$. Using Chow's lemma and taking projective closure, we may assume that (i) $X \to S$ is projective; blowing up Z we may assume that (ii) Z is the support of a divisor in X.

8.4. We extend the field K, i.e. we take base change with $S' \to S$ finite as in 6.8 to arrive at the situation where

(iii) X_η is geometrically reduced and irreducible.

8.5. Thus for some $U \subset S$ open nonempty we have that $\mathrm{sm}(X/S)$ is dense in all fibres over U and $\mathrm{sm}(X/S)_u$ is geometrically irreducible. Let $S \backslash U = \{ s_1, \ldots, s_r \}$. Choose for each $i \in \{ 1, \ldots, r \}$ a finite extension of complete discrete valuation rings $R_i = \mathcal{O}^{\wedge}_{S, s_i} \subset R'_i$ such that the normalization of $X \otimes R'_i$ has generically reduced special fibre (compare with 6.11). Since K is a global field, there exists a finite normal extension $K \subset K'$ such that R'_i embeds into $R_i \otimes_R K'$ for all $i = 1, \ldots, r$. Thus if we take

$S' = \operatorname{Spec} R'$ and X' equal to the normalization of $X \otimes R'$, then we see that we reduce to the case where we have

(v) $\mathrm{sm}(X/S)$ is dense in all fibres of $X \to S$.

8.6. Let $U \subset S$ be the open subscheme of 8.5. First we replace K by a finite extension and S by S′ such that all geometric components of X_s, for $s \in S \backslash U$, are defined over $\kappa(s)$. For any $s \in S \backslash U$ and any component C of X_s, we can find a finite extension $S' \to S$ and a section $\sigma : S' \to \mathrm{sm}(X/S)$ such that $\sigma(S'_s) \subset C$. This follows from Rumely's theorem (see [17, Theorem 1.7]), applied to the scheme $\mathrm{sm}(X/S) \backslash (X_s \backslash C)$ over S. We may also avoid any given number of multisections σ_α. Replacing S by a finite extension we may assume that we have mutually disjoint sections $\sigma_1, \ldots, \sigma_r : S \to X$ into the smooth locus of X over S, such that for all $s \in S$, and any component C of X_s, there is an i with $\sigma_i(s) \in C$.

Take a closed immersion $X \to \mathbf{P}^N_S$. Let $d = \dim X/S$. For each $i = 1, \ldots, r$ we put $L_i \subset \mathbf{P}^N_S$ equal to the tangent space of X at σ_i, seen as a linear subspace of relative dimension d over S of $\mathbf{P}^N_S$. Next, let $G \to S$ be the Grassmanian of $(N - d - 1)$-dimensional linear subvarieties L of $\mathbf{P}^N_S$. There is an open subscheme $V \subset G$ consisting of those L such that $L \cap X = \emptyset$ and $L \cap L_i = \emptyset$. Note that $V \to S$ is surjective. Thus, using Rumely's theorem, after replacing S by a finite extension, we can find such an L defined over S. We consider the projection morphism

$$\mathrm{pr}_L : X \to \mathbf{P}(L),$$

where $\mathbf{P}(L) \cong \mathbf{P}^d_S$ denotes the space of $(N - d)$-planes through L, and a point $x \in X$ maps to the $(N - d)$-plane that contains x and L. By our choice of L we see that pr_L is étale along the sections $\sigma_1, \ldots, \sigma_r$. We conclude that we may assume:

(v)′ There exists a finite morphism $\pi : X \to \mathbf{P}^d_S$ étale over an open subscheme $V \subset \mathbf{P}^d_S$ which surjects onto S.

8.7. Once again using Rumely's theorem, we may assume that there is a section $\sigma : S \to V$, with $V \subset \mathbf{P}^d_S$ as in (v)′. Applying the arguments of 6.13 we get:

(vi) There exists a morphism $f : X \to Y$ of projective S-varieties having the following two properties:

(vi) *a)* All fibres of f are nonempty and equidimensional of dimension 1.

(vi) *b)* The smooth locus is dense in all fibres of f.

8.8. The rest of the argument works exactly as in 6.14-6.16. This ends the proof of Theorem 8.2.

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